THE MINIMUM DEGREE THRESHOLD FOR PERFECT GRAPH PACKINGS

DANIELA KÜHN AND DERYK OSTHUS

ABSTRACT. Let H be any graph. We determine up to an additive constant the minimum degree of a graph G which ensures that G has a perfect H-packing (also called an H-factor). More precisely, let $\delta(H,n)$ denote the smallest integer k such that every graph G whose order n is divisible by $|H|$ and with $\delta(G)\ge k$ contains a perfect H-packing. We show that

$$\delta(H,n)=\left(1-\frac{1}{\chi^*(H)}\right)n+O(1).$$

The value of $\chi^*(H)$ depends on the relative sizes of the colour classes in the optimal colourings of H and satisfies $\chi(H)-1<\chi^*(H)\le\chi(H)$.

1. Introduction

1.1. **Background.** Given two graphs H and G, an *H-packing in G* is a collection of vertex-disjoint copies of H in G. H-packings are natural generalizations of graph matchings (which correspond to the case when H consists of a single edge). An H-packing in G is called *perfect* if it covers all vertices of G. In this case, we also say that G contains an *H-factor* or a *perfect H-matching*. If H has a component which contains at least 3 vertices then the question whether G has a perfect H-packing is difficult from both a structural and algorithmic point of view: Tutte's theorem characterizes those graphs which have a perfect H-packing if H is an edge but for other graphs H no such characterization exists. Moreover, Hell and Kirkpatrick [11] showed that the decision problem whether a graph G has a perfect H-packing is NP-complete if and only if H has a component which contains at least 3 vertices. They were motivated by questions arising in timetabling (see [10]).

This leads to the search for simple sufficient conditions which ensure the existence of a perfect H-packing. A fundamental result of this kind is the theorem of Hajnal and Szemerédi [9] which states that every graph G whose order n is divisible by r and whose minimum degree is at least $(1-1/r)n$ contains a perfect K_r-packing. The minimum degree condition is easily seen to be best possible. (The case when $r=3$ was proved earlier by Corrádi and Hajnal [7].) The following result is a generalization of this to arbitrary graphs H.

Theorem 1. **[Komlós, Sárközy and Szemerédi** [17]**]** *For every graph H there exists a constant $C=C(H)$ such that every graph G whose order n is divisible by $|H|$ and whose minimum degree is at least $(1-1/\chi(H))n+C$ contains a perfect H-packing.*

This confirmed a conjecture of Alon and Yuster [3], who had obtained the above result with an additional error term of εn in the minimum degree condition. As observed in [3], there are graphs H for which the above constant C cannot be omitted completely. Thus one might think that this settles the question of which minimum degree guarantees a perfect H-packing.

However, there are graphs H for which the bound on the minimum degree can be improved significantly: Kawarabayashi [13] conjectured that if $H=K_\ell^-$ (i.e. a complete

graph with one edge removed) and $\ell \geq 4$ then one can replace the chromatic number with the critical chromatic number in Theorem 1 and take $C = 0$. He [13] proved the case $\ell = 4$ and together with Cooley, we proved the general case for all graphs whose order n is sufficiently large [6]. Here the *critical chromatic number* $\chi_{cr}(H)$ of a graph H is defined as $(\chi(H) - 1)|H|/(|H| - \sigma(H))$, where $\sigma(H)$ denotes the minimum size of the smallest colour class in a colouring of H with $\chi(H)$ colours. Note that $\chi_{cr}(H)$ always satisfies $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$ and equals $\chi(H)$ if and only if for every colouring of H with $\chi(H)$ colours all the colour classes have equal size.

The critical chromatic number was introduced by Komlós [15]. He (and independently Alon and Fischer [2]) observed that for *any* graph H it gives a lower bound on the minimum degree that guarantees a perfect H-packing.

Proposition 2. *For every graph H and every integer n that is divisible by $|H|$ there exists a graph G of order n and minimum degree $\lceil (1 - 1/\chi_{cr}(H))n \rceil - 1$ which does not contain a perfect H-packing.*

Komlós also showed that the critical chromatic number is the parameter which governs the existence of *almost* perfect packings in graphs of large minimum degree.

Theorem 3. **[Komlós** [15]**]** *For every graph H and every $\gamma > 0$ there exists an integer $n_1 = n_1(\gamma, H)$ such that every graph G of order $n \geq n_1$ and minimum degree at least $(1 - 1/\chi_{cr}(H))n$ contains an H-packing which covers all but at most γn vertices of G.*

Confirming a conjecture of Komlós [15], Shokoufandeh and Zhao [21] proved that the number of uncovered vertices can be reduced to a constant depending only on H.

1.2. **Main result.** Our main result is that for any graph H, either its critical chromatic number or its chromatic number is the relevant parameter which governs the existence of perfect packings in graphs of large minimum degree. The exact classification depends on a parameter which we call the highest common factor of H and which is defined as follows.

We say that a colouring of H is *optimal* if it uses exactly $\chi(H) =: \ell$ colours. Given an optimal colouring c, let $x_1 \leq x_2 \leq \cdots \leq x_\ell$ denote the sizes of the colour classes of c. Put $\mathcal{D}(c) := \{x_{i+1} - x_i \,|\, i = 1, \ldots, \ell - 1\}$. Let $\mathcal{D}(H)$ denote the union of all the sets $\mathcal{D}(c)$ taken over all optimal colouring c. We denote by $\mathrm{hcf}_\chi(H)$ the highest common factor of all integers in $\mathcal{D}(H)$. (If $\mathcal{D}(H) = \{0\}$ we set $\mathrm{hcf}_\chi(H) := \infty$.) We write $\mathrm{hcf}_c(H)$ for the highest common factor of all the orders of components of H. If $\chi(H) \neq 2$ we say that $\mathrm{hcf}(H) = 1$ if $\mathrm{hcf}_\chi(H) = 1$. If $\chi(H) = 2$ then we say that $\mathrm{hcf}(H) = 1$ if both $\mathrm{hcf}_c(H) = 1$ and $\mathrm{hcf}_\chi(H) \leq 2$. The following table gives some examples:

H	$\chi(H)$	$\chi_{cr}(H)$	$\mathrm{hcf}_\chi(H)$	$\mathrm{hcf}_c(H)$	$\mathrm{hcf}(H)$
C_{2k+1} $(k \geq 2)$	3	$2 + 1/k$	1	–	1
C_{2k}	2	2	∞	$2k$	$\neq 1$
$K_{1,2} \cup C_6$	2	9/5	1	3	$\neq 1$
$K_{1,4} \cup C_4$	2	3/2	3	1	$\neq 1$
$K_{1,2} \cup K_{1,4}$	2	4/3	2	1	1

Note that if all the optimal colourings of H have the property that all colour classes have equal size, then $\mathcal{D}(H) = \{0\}$ and so $\mathrm{hcf}(H) \neq 1$ in this case. So if $\chi_{cr}(H) = \chi(H)$, then $\mathrm{hcf}(H) \neq 1$. Moreover, it is easy to see that there are graphs H with $\mathrm{hcf}_\chi(H) = 1$ but such that for all optimal colourings c of H the highest common factor of all integers in $\mathcal{D}(c)$ is strictly bigger than one (for example, take H to be the graph obtained from $K_{1,4,6}$

by adding a new vertex and joining it to all the vertices in the vertex class of size 4). Thus for such graphs H we do need to consider all optimal colourings of H.

As indicated above, our main result is that in Theorem 1 one can replace the chromatic number by the critical chromatic number if $\mathrm{hcf}(H) = 1$.

Theorem 4. *Suppose that H is a graph with $\mathrm{hcf}(H) = 1$. Then there exists a constant $C = C(H)$ such that every graph G whose order n is divisible by $|H|$ and whose minimum degree is at least $(1 - 1/\chi_{cr}(H))n + C$ contains a perfect H-packing.*

Note that Proposition 2 shows the result is best possible up to the value of the constant C. A simple modification of the examples in [2, 15] shows that there are graphs H for which the constant C cannot be omitted entirely. Moreover, it turns out that Theorem 1 is already best possible up to the value of the constant C if $\mathrm{hcf}(H) \neq 1$ (see Propositions 6 and 7 for the details). In [19] we sketched a simpler argument that yields a weaker result than Theorem 4: there H had to be either connected or non-bipartite and we needed an additional error term of εn in the minimum degree condition.

If we combine Theorems 1 and 4 together with Propositions 2, 6 and 7 we obtain the statement indicated in the abstract. Let

$$\chi^*(H) := \begin{cases} \chi_{cr}(H) & \text{if } \mathrm{hcf}(H) = 1; \\ \chi(H) & \text{otherwise.} \end{cases}$$

Also let $\delta(H, n)$ denote the smallest integer k such that every graph G whose order n is divisible by $|H|$ and with $\delta(G) \geq k$ contains a perfect H-packing.

Theorem 5. *For every graph H there exists a constant $C = C(H)$ such that*

$$\left(1 - \frac{1}{\chi^*(H)}\right) n - 1 \leq \delta(H, n) \leq \left(1 - \frac{1}{\chi^*(H)}\right) n + C.$$

(The -1 on the left hand side can be omitted if $\mathrm{hcf}(H) = 1$ or if $\chi(H) \geq 3$.) Thus for perfect packings in graphs of large minimum degree, the parameter $\chi^*(H)$ is the relevant parameter, whereas for almost perfect packings it is $\chi_{cr}(H)$. Note that while the definition of the parameter χ^* is somewhat complicated, the form of Theorem 5 is exactly analogous to that of the Erdős-Stone theorem (see e.g. [8, Thm 7.1.2] or [4, Ch. IV, Thm. 20]), which implies that

$$ex(H, n) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) n, \tag{1}$$

where $ex(H, n)$ denotes the smallest number k such that every graph G of order n and average degree $> k$ contains a copy of H.

1.3. **Open problems.** Our constant C appearing in Theorems 4 and 5 is rather large since it is related to the number of partition classes (clusters) obtained by the Regularity lemma. It would be interesting to know whether one can take e.g. $C = |H|$. Another open problem is to characterize all those graphs H for which $\delta(H, n) = \lceil (1 - 1/\chi^*(H))n \rceil$. This is known to be the case e.g. for complete graphs [9] and, if n is large, for cycles [1] and for the case when $H = K_\ell^-$ [6]. Further observations on this problem can be found in [6] and [5].

1.4. **Algorithmic aspects.** Kann [12] showed that the optimization problem of finding a maximum H-packing is APX-complete if H is connected and $|H| \geq 3$ (i.e. it is not possible to approximate the optimum solution within an arbitrary factor unless P=NP). For such H and any $\gamma > 0$, we gave a polynomial time algorithm in [19] which finds a perfect H-packing if $\delta(G) \geq (1 - 1/\chi^*(H) + \gamma)n$. Also note that Theorem 4 immediately implies that the decision problem whether a graph G has a perfect H-packing is trivially solvable in polynomial time in this case. On the other hand, in [19] we showed that for many graphs H, the problem becomes NP-complete when the input graphs are all those graphs G with minimum degree at least $(1 - 1/\chi^*(H) - \gamma)|G|$, where $\gamma > 0$ is arbitrary. We were able to show this if H is complete or H is a complete ℓ-partite graph where all colour classes contain at least two vertices. It would certainly be interesting to know whether this extends to all graphs H which have a component with at least three vertices.

1.5. **Organization of the paper.** In the next section we introduce some basic definitions and then describe the extremal examples which show that our main result is best possible. In Section 3 we then state the Regularity lemma of Szemerédi and the Blow-up lemma of Komlós, Sárközy and Szemerédi. In Section 4 we give a rough outline of the structure of the proof and state some of the main lemmas. In Section 5 we consider the case where G is not similar to an extremal graph. In Section 6 we investigate perfect H-packings in complete ℓ-partite graphs (these results are needed when we apply the Blow-up lemma in Sections 5 and 7). In Section 7 we consider the case where G is similar to an extremal graph. Finally, we combine the results of the previous sections in Section 8 to prove Theorem 4. Our proof of Theorem 4 uses ideas from [17].

2. Notation and extremal examples

Throughout this paper we omit floors and ceilings whenever this does not affect the argument. We write $e(G)$ for the number of edges of a graph G, $|G|$ for its order, $\delta(G)$ for its minimum degree, $\Delta(G)$ for its maximum degree, $\chi(G)$ for its chromatic number and $\chi_{cr}(G)$ for its critical chromatic number as defined in Section 1. We denote the degree of a vertex $x \in G$ by $d_G(x)$ and its neighbourhood by $N_G(x)$. Given a set $A \subseteq V(G)$, we write $e(A)$ for the number of edges in A and define the density of A by $d(A) := e(A)/\binom{|A|}{2}$.

Given disjoint $A, B \subseteq V(G)$, an *A–B edge* is an edge of G with one endvertex in A and the other in B; the number of these edges is denoted by $e_G(A, B)$ or $e(A, B)$ if this is unambiguous. We write $(A, B)_G$ for the bipartite subgraph of G whose vertex classes are A and B and whose edges are all A–B edges in G. More generally, we write (A, B) for a bipartite graph with vertex classes A and B.

The following two propositions together show that if $\mathrm{hcf}(H) \neq 1$ then Theorem 1 is best possible up the the value of the constant C. Thus in this case the chromatic number of H is the relevant parameter which governs the existence of perfect matchings in graphs of large minimum degree. The first proposition deals with the case when $\chi(H) \geq 3$ as well as the case when $\chi(H) = 2$ and $\mathrm{hcf}_\chi(H) \geq 3$.

Proposition 6. *Let H be a graph with $2 \leq \chi(H) =: \ell$ and let $k \in \mathbb{N}$. Let G_1 be the complete ℓ-partite graph of order $k|H|$ whose vertex classes $U_1, \dots, U_\ell$ satisfy $|U_1| = \lfloor k|H|/\ell \rfloor + 1$, $|U_2| = \lceil k|H|/\ell \rceil - 1$ and $\lfloor k|H|/\ell \rfloor \leq |U_i| \leq \lceil k|H|/\ell \rceil$ for all $i \geq 3$. (So $\delta(G_1) = \lceil (1 - 1/\chi(H))|G_1| \rceil - 1$.) If $\ell \geq 3$ and $\mathrm{hcf}_\chi(H) \neq 1$ or if $\ell = 2$ and $\mathrm{hcf}_\chi(H) \geq 3$ then G_1 does not contain a perfect H-packing.*

Proof. Suppose first that $\ell \geq 3$ and that $\mathrm{hcf}_\chi(H)$ is finite. In this case there are vertex classes U_{i_1} and U_{i_2} such that $|U_{i_1}| - |U_{i_2}| = 1$ (note that these are not necessarily U_1

and U_2). Consider any H-packing in G_1 consisting of $H_1, \dots, H_r$ say. By induction one can show that $|U_{i_1} \setminus (H_1 \cup \cdots \cup H_r)| - |U_{i_2} \setminus (H_1 \cup \cdots \cup H_r)| \equiv 1 \mod \mathrm{hcf}_\chi(H)$. But as $\mathrm{hcf}_\chi(H) \neq 1$ this implies that at least one of $U_{i_1} \setminus (H_1 \cup \cdots \cup H_r)$ and $U_{i_2} \setminus (H_1 \cup \cdots \cup H_r)$ has to be non-empty. Thus the H-packing $H_1, \dots, H_r$ cannot be perfect. If $\ell \geq 3$ but $\mathrm{hcf}_\chi(H) = \infty$ then the colour classes of H have the same size and thus G_1 also works.

The case when $\ell = 2$ is similar except that we have to work with U_1 and U_2 and can only assume that $|U_1| - |U_2| \in \{1, 2\}$. □

In the next proposition we consider the case when $\chi(H) = 2$ and $\mathrm{hcf}_c(H) \neq 1$. We omit its proof as it is similar to the proof of Proposition 6.

Proposition 7. *Let H be a bipartite graph with $\mathrm{hcf}_c(H) \neq 1$ and let $k \in \mathbb{N}$. If $\mathrm{hcf}_c(H) = 2$ and $k|H|$ is not divisible by 4 let G_2 be the disjoint union of two cliques of order $k|H|/2$. Otherwise let G_2 be the disjoint union of two cliques of orders $\lfloor k|H|/2 \rfloor + 1$ and $\lceil k|H|/2 \rceil - 1$. (So $\delta(G_2) \geq (1 - 1/\chi(H))|G_1| - 2$.) G_2 does not contain a perfect H-packing.* □

The following corollary gives a characterization of those graphs with $\mathrm{hcf}(H) = 1$. It follows immediately from Propositions 6 and 7 as well as Lemmas 15–17 in Section 6. We will not need it in the proof of Theorem 4 but state it as the characterization may be of independent interest.

Corollary 8. *Let H be a graph with $2 \leq \chi(H) =: \ell$. Let $k' \gg |H|$ be an integer and let G_1 and G_2 be the graphs defined in Propositions 6 and 7 for $k := \ell k'$. If $\chi(H) \geq 3$ then G_1 contains a perfect H-packing if and only if $\mathrm{hcf}(H) = 1$. Similarly, if $\chi(H) = 2$ then both G_1 and G_2 contain a perfect H-packing if and only if $\mathrm{hcf}(H) = 1$.* □

3. The Regularity lemma and the Blow-up lemma

The purpose of this section is to collect all the information we need about the Regularity lemma and the Blow-up lemma. See [18] and [14] for surveys about these. Let us start with some more notation. The *density* of a bipartite graph $G = (A, B)$ is defined to be

$$d_G(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

We also write $d(A, B)$ if this is unambiguous. Given $\varepsilon > 0$, we say that G is *ε-regular* if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $|d(A, B) - d(X, Y)| < \varepsilon$. Given $d \in [0, 1]$, we say that G is *(ε, d)-superregular* if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $d(X, Y) > d$ and, furthermore, if $d_G(a) > d|B|$ for all $a \in A$ and $d_G(b) > d|A|$ for all $b \in B$.

We will use the following degree form of Szemerédi's Regularity lemma which can be easily derived from the classical version. Proofs of the latter are for example included in [4] and [8].

Lemma 9 (Regularity lemma)**.** *For all $\varepsilon > 0$ and all integers k_0 there is an $N = N(\varepsilon, k_0)$ such that for every number $d \in [0, 1]$ and for every graph G on at least N vertices there exist a partition of $V(G)$ into $V_0, V_1, \dots, V_k$ and a spanning subgraph G' of G such that the following holds:*

- *$k_0 \leq k \leq N$,*
- *$|V_0| \leq \varepsilon|G|$,*
- *$|V_1| = \cdots = |V_k| =: L$,*
- *$d_{G'}(x) > d_G(x) - (d + \varepsilon)|G|$ for all vertices $x \in G$,*
- *for all $i \geq 1$ the graph $G'[V_i]$ is empty,*

- *for all* $1 \le i < j \le k$ *the graph* $(V_i, V_j)_{G'}$ *is* ε*-regular and has density either* 0 *or* $> d$.

The sets V_i ($i \ge 1$) are called *clusters*, V_0 is called the *exceptional set*. Given clusters and G' as in Lemma 9, the *reduced graph* R is the graph whose vertices are $V_1, \dots, V_k$ and in which V_i is joined to V_j whenever $(V_i, V_j)_{G'}$ is ε-regular and has density $> d$. Thus V_iV_j is an edge of R if and only if G' has an edge between V_i and V_j.

Given a set $A \subseteq V(R)$, we call the set of all those vertices of G which are contained in clusters belonging to A the *blow-up of* A. Similarly, if R' is a subgraph of R, then the *blow-up of* R' is the subgraph of G' induced by the blow-up of $V(R')$.

We will also use the Blow-up lemma of Komlós, Sárközy and Szemerédi [16]. It implies that dense regular pairs behave like complete bipartite graphs with respect to containing bounded degree graphs as subgraphs.

Lemma 10 (Blow-up lemma)**.** *Given a graph* F *on* $\{1, \dots, f\}$ *and positive numbers* d, Δ*, there is a positive number* $\varepsilon_0 = \varepsilon_0(d, \Delta, f)$ *such that the following holds. Given* $L_1, \dots, L_f \in \mathbb{N}$ *and* $\varepsilon \le \varepsilon_0$*, let* F^* *be the graph obtained from* F *by replacing each vertex* $i \in F$ *with a set* V_i *of* L_i *new vertices and joining all vertices in* V_i *to all vertices in* V_j *whenever* ij *is an edge of* F*. Let* G *be a spanning subgraph of* F^* *such that for every edge* $ij \in F$ *the graph* $(V_i, V_j)_G$ *is* (ε, d)*-superregular. Then* G *contains a copy of every subgraph* H *of* F^* *with* $\Delta(H) \le \Delta$.

4. Preliminaries and overview of the proof

Let H be a graph of chromatic number $\ell \ge 2$. Put

$$z_1 := (\ell - 1)\sigma(H), \quad z := |H| - \sigma(H), \quad \xi := \frac{z_1}{z} = \frac{(\ell - 1)\sigma(H)}{|H| - \sigma(H)}. \tag{2}$$

(Recall that $\sigma(H)$ is the smallest colour class in any ℓ-colouring of H.) Note that $\xi < 1$ if $\chi_{cr}(H) < \chi(H)$ and so in particular if $\mathrm{hcf}(H) = 1$. Let B^* denote the complete ℓ-partite graph with one vertex class of size z_1 and $\ell - 1$ vertex classes of size z. Note that B^* has a perfect H-packing consisting of $\ell - 1$ copies of H. Moreover, it is easy to check that

$$\chi_{cr}(H) = \chi_{cr}(B^*) = \ell - 1 + \xi. \tag{3}$$

Call B^* the *bottlegraph assigned to* H. We now give an overview of the proof of Theorem 4. In Section 5.1 we first apply the Regularity lemma to G in order to obtain a set V_0 of exceptional vertices and a reduced graph R. It will turn out that the minimum degree of R is almost $(1 - 1/\chi_{cr}(B^*))|R|$. So R has an almost perfect B^*-packing $\mathcal{B}'$ by Theorem 3. Let $B_1, \dots, B_{k'}$ denote the copies of B^* in $\mathcal{B}'$. Our aim in Sections 5.2 and 5.3 is to show that one can take out a small number of suitably chosen copies of B^* from G to achieve that the following conditions hold:

(α) Each vertex in V_0 lies in one of these copies of B^* taken out from G. Moreover, each vertex that does not belong to a blow-up of some $B_t \in \mathcal{B}'$ also lies in one of these copies of B^*.

(β) The (modified) blow-up of each $B_t \in \mathcal{B}'$ has a perfect H-packing.

Note that if we say that we take out a copy of B^* (or H) from G then we mean that we delete all its vertices from G and thus also from the clusters they belong to. So in particular the (modified) blow-up of B_t no longer contains these vertices.

For all $t \le k'$ and all $j \le \ell$ let $X_j(t)$ denote the (modified) blow-up of the jth vertex class of $B_t \in \mathcal{B}'$ (where the ℓth vertex class of B_t is the small one). It will turn out that (β) holds if the $X_j(t)$ satisfy the conditions in the following definition. (The graph

$G' \subseteq G$ obtained from the Regularity lemma will play the role of G^* in Definition 11. So it will be easy to satisfy condition (a) of Definition 11.)

Definition 11. Suppose that G is a graph whose order n is divisible by $|B^*|$. Let $k \geq 1$ be an integer and let $\varepsilon \ll d \ll \beta \ll 1$ be positive constants. We say that G *has a blown-up B^*-cover for parameters ε, d, β, k* if there exists a spanning subgraph G^* of G and a partition $X_1(1), \dots, X_\ell(1), \dots, X_1(k), \dots, X_\ell(k)$ of the vertex set of G such that the following holds:

(a) All the bipartite subgraphs $(X_j(t), X_{j'}(t))_{G^*}$ of G^* between $X_j(t)$ and $X_{j'}(t)$ are (ε, d)-superregular whenever $j \neq j'$.
(b) $|X_1(t) \cup \dots \cup X_\ell(t)|$ is divisible by $|B^*|$ for all $t \leq k$.
(c) $(1-\beta^{1/10})|X_\ell(t)| \leq \xi|X_j(t)| \leq (1-\beta)|X_\ell(t)|$ for all $j < \ell$ and all $t \leq k$ and $|\,|X_j(t)| - |X_{j'}(t)|\,| \leq d|X_1(t) \cup \dots \cup X_\ell(t)|$ for all $1 \leq j < j' < \ell$ and all $t \leq k$.

For all $t \leq k$, we call the ℓ-partite subgraph of G^* whose vertex classes are the sets $X_1(t), \dots, X_\ell(t)$ the *t'th element* of the blown-up B^*-cover. The *complete ℓ-partite graph corresponding to the t'th element* is the one whose vertex classes have sizes $|X_1(t)|, \dots, |X_\ell(t)|$. Note that condition (c) implies that for all $j < \ell$ the ratio of $|X_j(t)|$ to the total size of the tth element is a little smaller than $z/|B^*|$ (recall that z is the size of the large vertex classes of B^*).

The following lemma implies that the complete ℓ-partite graph corresponding to some element of a blown-up B^*-cover contains a perfect H-packing. Combined with the Blow-up lemma, this will imply that each element of the blown-up B^*-cover has a perfect H-packing. (Thus (β) will be satisfied if the $X_j(t)$ are as in Definition 11.)

Lemma 12. *Let H be a graph with $\ell := \chi(H) \geq 2$ and $\mathrm{hcf}(H) = 1$. Let ξ be as defined in (2). Let $0 < d \ll \beta \ll \xi, 1-\xi, 1/|H|$ be positive constants. Suppose that F is a complete ℓ-partite graph with vertex classes $U_1, \dots, U_\ell$ such that $|F| \gg |H|$ is divisible by $|H|$, $(1-\beta^{1/10})|U_\ell| \leq \xi|U_i| \leq (1-\beta)|U_\ell|$ for all $i < \ell$ and such that $|\,|U_i| - |U_j|\,| \leq d|F|$ whenever $1 \leq i < j < \ell$. Then F contains a perfect H-packing.*

Lemma 12 will be proved at the end of Section 6, where we will deduce it from Lemmas 18 and 19, which are also proved in that section. Lemma 12 is one of the points where the condition that $\mathrm{hcf}(H) = 1$ is necessary.

The following lemma shows that we can find a blown-up B^*-cover as long as G satisfies certain properties. Roughly speaking these properties (i) and (ii) say that G is not too close to being one of the extremal graphs having minimum degree almost $(1 - 1/\chi_{cr}(H))n$ but not containing a perfect H-packing. We will refer to this as the non-extremal case.

Lemma 13. *Let H be a graph of chromatic number $\ell \geq 2$ such that $\chi_{cr}(H) < \ell$. Let B^* denote the bottlegraph assigned to H and let z and ξ be as defined in (2). Let*

$$\varepsilon' \ll d' \ll \theta \ll \tau \ll \xi, 1-\xi, 1/|B^*|$$

be positive constants. There exist integers n_0 and $k_1 = k_1(\varepsilon', \theta, B^*)$ such that the following holds. Suppose that G is a graph whose order $n \geq n_0$ is divisible by $|B^*|$ and whose minimum degree satisfies $\delta(G) \geq (1 - \frac{1}{\chi_{cr}(H)} - \theta)n$. Furthermore, suppose that G satisfies the following further properties:

(i) *G does not contain a vertex set A of size $zn/|B^*|$ such that $d(A) \leq \tau$.*
(ii) *Additionally, if $\ell = 2$ then G does not contain a vertex set A such that $d(A, V(G) \setminus A) \leq \tau$.*

Then there exists a family $\mathcal{B}^*$ of at most $\theta^{1/3}n$ disjoint copies of B^* in G such that the graph $G - \bigcup \mathcal{B}^*$ (which is obtained from G by taking out all the copies of B^* in $\mathcal{B}^*$) has a blown-up B^*-cover with parameters $2\varepsilon', d'/2, 2\theta, k_1$.

Lemma 13 will be proved in Section 5. Lemmas 12 and 13 together with the Blow-up lemma imply that in the non-extremal case we can satisfy conditions (α) and (β), i.e. we have a perfect H-packing in this case. This is formalized in the following corollary.

Corollary 14. *Let H be a graph of chromatic number $\ell \geq 2$ such that $\mathrm{hcf}(H) = 1$. Let B^* denote the bottlegraph assigned to H and let z and ξ be as defined in (2). Let $\theta \ll \tau \ll \xi, 1 - \xi, 1/|B^*|$ be positive constants. There exists an integer n_0 such that the following holds. Suppose that G is a graph whose order $n \geq n_0$ is divisible by $|B^*|$ and whose minimum degree satisfies $\delta(G) \geq (1 - \frac{1}{\chi_{cr}(H)} - \theta)n$. Furthermore, suppose that G satisfies the following further properties:*

(i) *G does not contain a vertex set A of size $zn/|B^*|$ such that $d(A) \leq \tau$.*
(ii) *Additionally, if $\ell = 2$ then G does not contain a vertex set A such that $d(A, G - A) \leq \tau$.*

Then G has a perfect H-packing.

Proof of Corollary 14. Fix positive constants ε', d' such that

$$\varepsilon' \ll d' \ll \theta \ll \tau \ll \xi, 1 - \xi, 1/|B^*|.$$

An application of Lemma 13 shows that by taking out a small number of disjoint copies of B^* from G we obtain a subgraph which has a blown-up B^*-cover with parameters $2\varepsilon', d'/2, 2\theta, k_1$. Conditions (b) and (c) in Definition 11 imply that the complete ℓ-partite graphs corresponding to the k_1 elements of this blown-up B^*-cover satisfy the assumptions of Lemma 12 with $d := d'/2$ and where 2θ plays the role of β. Thus each of these complete ℓ-partite graphs contains a perfect H-packing. Condition (a) in Definition 11 ensures that we can now apply the Blow-up lemma (Lemma 10) to each of the k_1 elements in the blown-up B^*-cover to obtain a perfect H-packing of this element. All these H-packings together with the copies of B^* taken out earlier in order to obtain the blown-up B^*-cover yield a perfect H-packing of G. □

The extremal cases (i.e. where G satisfies either (i) or (ii)) will be dealt with in Section 7. These cases also rely on Lemma 13. For example if G satisfies (i) but $G - A$ does not satisfy (i) or (ii), then very roughly the strategy is to apply Lemma 13 to find a perfect B_1^*-packing of $G - A$, where B_1^* is obtained from B^* by removing one of the large colour classes. The minimum degree of G will ensure that the bipartite subgraph spanned by A and $V(G) - A$ is almost complete. This will be used to extend the B_1^*-packing of $G - A$ to a perfect B^*-packing of G. The reason we considered sets A of size $zn/|B^*|$ in (i) is that this is precisely the number of vertices needed to extend each copy of B_1^* to a copy of B^*. (Recall that z was the size of the large vertex classes of B^*.)

However, for this strategy to work, we first need to modify the set A slighly. We will also take out some carefully chosen copies of H from G. One matter which complicates the argument is that B_1^* does not necessarily satisfy $\mathrm{hcf}(B_1^*) = 1$. This means that we cannot find perfect B_1^*-packing of $G - A$ by a direct application of Lemma 13, as the blown-up B_1^*-cover produced by that lemma does not necessarily yield a perfect B_1^*-packing of $G - A$. To overcome this difficulty, we will work directly with the blown-up B_1^*-cover. So the use of Lemma 13 in Section 7 is the reason why we do not assume $\mathrm{hcf}(H) = 1$ in Lemma 13. It is also the reason why we allow for an error term θn in the minimum degree condition on G.

5. The non-extremal case: proof of Lemma 13

The purpose of this section is to prove Lemma 13.

5.1. **Applying the Regularity lemma and choosing a packing of the reduced graph.** We will fix further constants satisfying the following hierarchy

$$0 < \varepsilon \ll \varepsilon' \ll d' \ll d \ll \theta \ll \tau \ll \xi, 1-\xi, 1/|B^*|. \tag{4}$$

Moreover, we choose an integer k_0 such that

$$k_0 \geq n_1(\theta, B^*), \tag{5}$$

where n_1 is as defined in Theorem 3. We put

$$k_1 := \lfloor N(\varepsilon, k_0)/|B^*| \rfloor, \tag{6}$$

where $N(\varepsilon, k_0)$ is as defined in the Regularity lemma (Lemma 9). In what follows, we assume that the order n of our given graph G is sufficiently large for our estimates to hold. We now apply the Regularity lemma with parameters ε, d and k_0 to G to obtain clusters, an exceptional set V_0, a spanning subgraph $G' \subseteq G$ and a reduced graph R. (4) together with the well-known fact that the minimum degree of G is almost inherited by its reduced graph (see e.g. [20, Prop. 9] for an explicit proof) implies that

$$\delta(R) \geq \left(1 - \frac{1}{\chi_{cr}(H)} - 2\theta\right)|R| \overset{(3)}{=} \left(1 - \frac{1}{\ell - 1 + \xi} - 2\theta\right)|R|. \tag{7}$$

Since $|R| \geq k_0 \geq n_1(\theta, B^*)$ by (5), we may apply Theorem 3 to R to find a B^*-packing $\mathcal{B}'$ which covers all but at most $\sqrt{\theta}|R|$ vertices of R. (More precisely, we apply Theorem 3 to a graph R' which is obtained from R by adding at most $\theta^{3/4}|R|$ new vertices and connecting them to all other vertices. By (3) and (7), we have $\delta(R') \geq (1 - 1/\chi_{cr}(B^*))|R'|$, as required in Theorem 3. Removing the new vertices results in a B^*-packing of R which has the desired size.) We delete all the clusters not contained in some copy of B^* in $\mathcal{B}'$ from R and add all the vertices lying in these clusters to the exceptional set V_0. Thus $|V_0| \leq \varepsilon n + \sqrt{\theta}n \leq 2\sqrt{\theta}n$. From now on, we denote by R the subgraph of the reduced graph induced by all the remaining clusters. Thus $\mathcal{B}'$ now is a perfect B^*-packing of R and we still have that

$$\delta(R) \geq \left(1 - \frac{1}{\ell - 1 + \xi} - 2\sqrt{\theta}\right)|R|. \tag{8}$$

It is easy to check that for all $B \in \mathcal{B}'$ we can replace each cluster V_a in B by a subcluster of size $L' := (1 - \varepsilon|B^*|)L$ such that for each edge V_aV_b of B the bipartite subgraph of G' between the chosen subclusters of V_a and V_b is $(2\varepsilon, d/2)$-superregular (see e.g. [20, Prop. 8]). Add all the vertices of G which do not lie in one of the chosen subclusters to the exceptional set V_0. Then

$$|V_0| \leq 3\sqrt{\theta}n.$$

By adjusting L' if necessary and adding a bounded number of further vertices to V_0 we may assume that L' is divisible by z_1z. (Recall that z_1 and z were defined in (2).) From now on, we refer to the chosen subclusters as the clusters of R.

Next we partition each of these clusters V_a into a red part V_a^{red} and a blue part V_a^{blue} such that $|V_a^{red}| = \theta^2|V_a|$ and such that $\big||N_G(x) \cap V_a^{red}| - \theta^2|N_G(x) \cap V_a|\big| \leq \varepsilon L'$ for every vertex $x \in G$. (Consider a random partition to see that there are V_a^{red} and V_a^{blue} with these properties.) Together all these partitions of the clusters of R yield a partition of the vertices of $G - V_0$ into red and blue vertices. We will use these partitions to ensure that even after some modifications which we have to carry out during the proof, the edges of

the $B \in \mathcal{B}'$ will still correspond to superregular subgraphs of G'. More precisely, during the proof we will take out certain copies of B^* from G, but each copy will avoid all the red vertices. All the vertices contained in these copies of B^* will be removed from the clusters they belong to. However, if we look at the (modified) bipartite subgraph of G' which corresponds to some edge V_aV_b of $B \in \mathcal{B}'$, then this subgraph of G' will still be (ε', d')-superregular since it still contains all vertices in V_a^{red} and V_b^{red}.

The blown-up B^*-cover required in Lemma 13 will be otained from the cover corresponding to $\mathcal{B}'$ by taking out a small number of copies of B^* from G. This will be done in two steps. Firstly, we will take out copies of B^* to ensure that for every $B \in \mathcal{B}'$ the size of the blow-up of each of its $\ell - 1$ large vertex classes is significantly smaller than $(z/|B^*|)$times the size of the blow-up of the entire B. This will ensure that the cover corresponding to $\mathcal{B}'$ satisfies condition (c) in the definition of blown-up B^*-cover (Definition 11). Moreover, each exceptional vertex will be contained in one of the copies taken out. So after this step we also have incorporated all the exceptional vertices. All the copies of B^* deleted in this process will avoid the red vertices of G. In the second step we will then take out a bounded number of further copies of B^* in order to achieve that the blow-up of each $B \in \mathcal{B}'$ is divisible by $|B^*|$ (as required in condition (b) in Definition 11). As mentioned in the previous paragraph, the blown-up B^*-cover thus obtained from $\mathcal{B}'$ will also satisfy condition (a) in Definition 11 since in this last step we remove only a bounded number of further vertices from the clusters, which does not affect the superregularity significantly. Finally, since the blown-up B^* cover obtained in this way has $|\mathcal{B}'| \le k_1$ elements, we may have to split some of the blown up copies of B^* to obtain a blown-up B^* cover with exactly k_1 elements.

5.2. **Adjusting the sizes of the vertex classes in the blow-ups of the $B \in \mathcal{B}'$.** Let $B_1, \dots, B_{k'}$ denote the copies of B^* in $\mathcal{B}'$. As described at the end of the previous section, our next aim is to take out a small number of copies of B^* from G to achieve that, for all $t \le k'$, the blow-up of each large vertex class of B_t is significantly smaller than $(z/|B^*|)$times the size of the blow-up of B_t itself. It turns out that this becomes simpler if we first split the blow-up of each B_t into $z_1 z$ 'smaller blow-ups'. Then we take out copies of B^* from G in order to modify the sizes of these smaller blow-ups. This will imply that sizes of the blown-up vertex classes of the original B_t's are as desired. We will not remove red vertices in this process.

Thus consider any B_t. We will think of the ℓth vertex class of B_t as the one having size z_1. For all $j < \ell$, split each of the z clusters belonging to the jth vertex class of B_t into z_1 subclusters of equal size. Let $Z'_j((t-1)z_1 z + 1), \dots, Z'_j(tz_1 z)$ denote the subclusters thus obtained. Similarly, split each cluster belonging to the ℓth vertex class of B_t into z subclusters of equal size. Let $Z_\ell((t-1)z_1 z + 1), \dots, Z_\ell(tz_1 z)$ denote the subclusters thus obtained. Put

$$k'' := z_1 z k'.$$

Given $i \le k''$, we think of the ℓ-partite subgraph of G' with vertex classes $Z'_1(i), \dots, Z'_{\ell-1}(i), Z_\ell(i)$ as a blown-up copy of B^*. (Indeed, note that $\xi |Z'_j(i)| = |Z_\ell(i)|$ for all $j < \ell$.) We may assume that about $\theta^2 |Z'_j(i)|$ vertices in $|Z'_j(i)|$ are red and that for every vertex $x \in G$ about a θ^2-fraction of its neighbours in each $Z'_j(i)$ are red and that the analogue holds for $Z_\ell(i)$. (Indeed, consider random partitions again to show that this can be guaranteed.)

In order to achieve that the size of each large vertex class is significantly smaller than the size of the entire blown-up copy, we will remove a θ-fraction of vertices from each of $Z'_1(i), \dots, Z'_{\ell-1}(i)$. We will add all these vertices to V_0. The aim then is to incorporate

all the vertices in V_0 by taking out copies of B^* from G. Of course, this has to be done in such a way that we don't destroy the properties of the vertex classes again. So put

$$L'' := (1-\theta)L'/z_1.$$

For all $i \le k''$ and all $j < \ell$ remove $\theta L'/z_1$ blue vertices from $Z'_j(i)$ and add all these vertices to V_0. Denote the subset of $Z'_j(i)$ thus obtained by $Z_j(i)$. So

$$|Z_\ell(i)| = L'/z = \xi L''/(1-\theta), \quad |Z_j(i)| = L'' = (1-\theta)|Z_\ell(i)|/\xi \tag{9}$$

whenever $j < \ell$. Also, we now have that

$$|V_0| \le 4\sqrt{\theta}n. \tag{10}$$

Note that for all $i \le k''$ and all $0 \le j < j' \le \ell$ the graph $(Z_j(i), Z_{j'}(i))_{G'}$ is ε'-regular and has density at least d'. Denote by $Z_j^{red}(i)$ the set of red vertices in $Z_j(i)$.

Let us now prove the following claim. Roughly speaking, it states that by taking out a small number of copies of B^* from G we can incorporate all the vertices in V_0 and that this can be done without destroying the properties of the vertex classes of the blown-up copies of B^*.

Claim. *We can take out at most $3|V_0| \le 12\sqrt{\theta}n$ disjoint copies of B^* from G which cover all the vertices in V_0 and have the property that the leftover sets $Y_j(i) \subseteq Z_j(i)$ thus obtained satisfy*

(a) $Z_j^{red}(i) \subseteq Y_j(i)$,
(b) $L'' - |Y_1(i)| \le \theta^{1/7}L''$,
(c) $|Y_1(i)| = \cdots = |Y_{\ell-1}(i)| \le (1-\theta)|Y_\ell(i)|/\xi$.

To prove this claim, we show that for every vertex $x \in V_0$ in turn we can take out either one, two or three disjoint copies of B^* which satisfy the following three properties.

> *Firstly, x lies in one of the copies. Secondly, these copies avoid all the red vertices. Thirdly, when removing these copies from G then, for every $i \le k''$, we either delete no vertex at all in $Z_1(i) \cup \cdots \cup Z_\ell(i)$ or else we delete precisely z vertices in each of $Z_1(i), \ldots, Z_{\ell-1}(i)$ and delete either z_1 or $z_1 - 1$ vertices in $Z_\ell(i)$.* $(*)$

Together with (9) this implies that after each step the subsets obtained from the $Z_j(i)$ will satisfy conditions (a) and (c). We will discuss later how (b) can be satisfied too.

Thus consider the first vertex $x \in V_0$. To find the copies of B^* satisfying $(*)$ we will distinguish several cases. Suppose first that there exists an index $i = i(x)$ such that x has at least $\theta L''$ neighbours in $Z_j(i)$ for all $j < \ell$.[1] Take out a copy of B^* from G which contains x, which meets each of $Z_1(i), \ldots, Z_{\ell-1}(i)$ in z vertices and $Z_\ell(i)$ in $z_1 - 1$ vertices and which avoids the red vertices of G. (The existence of such copies of B^* in G easily follows from a 'greedy' argument based on the ε'-regularity of the bipartite subgraphs $(Z_j(i), Z_{j'}(i))_{G'}$ of G', see e.g. Lemma 7.5.2 in [8] or Theorem 2.1 in [18]. We will often use this and similar facts below. We can avoid the red vertices since $|Z_j^{red}(i)| \ll \theta L''$ and so most of the neighbours of x in $Z_j(i)$ will be blue.)

Next suppose that we cannot find an index i as above. By relabelling if necessary, we may assume that x has at most $\theta L''$ neighbours in $Z_1(i)$ for all $i \le k''$. Let I denote the set

[1] In later steps we will ask whether a vertex $x' \in V_0$ has at least $\theta L''$ neighbours in the *current* set $Z_j(i)$ for all $j < \ell$.

of all those indices i for which x has at least $\theta L''$ neighbours in $Z_j(i)$ for all $j = 2, \dots, \ell$. To obtain a lower bound on the size of I, we now consider $d_{G'}(x)$. This shows that

$$\begin{aligned}(k'' - |I|)L''(\ell - 2 + 2\theta) + |I|L''(\ell - 2 + \xi/(1-\theta) + \theta) &\overset{(9)}{\geq} d_{G'}(x) - |V_0| \\ &\overset{(10)}{\geq} (1 - 1/\chi_{cr}(H) - \theta^{1/3})n.\end{aligned}$$

The term $\theta^{1/3}n$ in the second line is a bound on $|V_0|$ (with room to spare – this will be useful later on). The above equation implies that

$$|I| \geq (1 - \theta^{1/4})k''. \tag{11}$$

Now suppose that there are two indices $i_1, i_2 \in I$ such that the density $d_{G'}(Z_1(i_1), Z_j(i_2))$ is nonzero for all $j = 1, \dots, \ell - 1$. (Note that the last condition in Lemma 9 implies that then each of the bipartite subgraphs $(Z_1(i_1), Z_j(i_2))_{G'}$ of G' is ε'-regular and has density at least d'.) In this case we take out two disjoint copies of B^*. The first contains x and has z vertices in each of $Z_2(i_1), \dots, Z_{\ell-1}(i_1)$, $z - 1$ vertices in $Z_1(i_1)$ and z_1 vertices in $Z_\ell(i_1)$. Such a copy of B^* exists since $i_1 \in I$. The second copy will have one vertex in $Z_1(i_1)$, z vertices in each of $Z_1(i_2), \dots, Z_{\ell-1}(i_2)$ and $z_1 - 1$ vertices in $Z_\ell(i_2)$. Again, these copies of B^* are chosen such that they avoid the red vertices of G. So we may assume that there are no indices i_1, i_2 as above.

Suppose next that there are indices $i_3, i_4 \in I$ and $j^* = j^*(i_3, i_4)$ with $2 \leq j^* \leq \ell - 1$ and such that $d_{G'}(Z_{j^*}(i_3), Z_j(i_4)) > 0$ for all $j = 2, \dots, \ell$ and $d_{G'}(Z_j(i_3), Z_1(i_4)) > 0$ for all $j^* \neq j \leq \ell$. In this case we take out 2 disjoint copies of B^* again. The first one contains x, has z vertices in $Z_{j^*}(i_3)$ and in each of $Z_2(i_4), \dots, Z_{\ell-1}(i_4)$ and $z_1 - 1$ vertices in $Z_\ell(i_4)$. The second copy will have z_1 vertices in $Z_\ell(i_3)$ and z vertices in $Z_1(i_4)$ as well as z vertices in each $Z_j(i_3)$ with $j^* \neq j < \ell$. Again, all these copies of B^* are chosen such that they avoid all the red vertices. So we may assume that there are no such indices i_3, i_4, j^*.

Suppose next that there are indices $i_5, i_6, i_7 \in I$ and $j^\diamond = j^\diamond(i_5, i_6, i_7)$ with $2 \leq j^\diamond \leq \ell - 1$ and such that $d_{G'}(Z_{j^\diamond}(i_5), Z_j(i_7)) > 0$ for all $j = 1, \dots, \ell - 1$ and $d_{G'}(Z_j(i_5), Z_1(i_6)) > 0$ for all $j^\diamond \neq j \leq \ell$. In this case we take out 3 disjoint copies of B^*. The first copy contains x, has $z - 1$ vertices in $Z_1(i_6)$, z vertices in each of $Z_2(i_6), \dots, Z_{\ell-1}(i_6)$ and z_1 vertices in $Z_\ell(i_6)$. The second copy has one vertex in $Z_1(i_6)$, $z - 1$ vertices in $Z_{j^\diamond}(i_5)$, z vertices in each $Z_j(i_5)$ with $j^\diamond \neq j < \ell$ and z_1 vertices in $Z_\ell(i_5)$. The third copy has one vertex in $Z_{j^\diamond}(i_5)$, z vertices in each of $Z_1(i_7), \dots, Z_{\ell-1}(i_7)$ and $z_1 - 1$ vertices in $Z_\ell(i_7)$. Again, all these copies of B^* are chosen such that they avoid all the red vertices. So we may assume that there are no such indices $i_5, i_6, i_7, j^\diamond$.

We will show that together with our previous three assumptions this leads to a contradiction to our assumption on the minimum degree of G. For this, first note that there are at least $\tau|I|^2/4$ ordered pairs of indices $i, i' \in I$ for which $d_{G'}(Z_1(i), Z_1(i')) > 0$. Indeed, otherwise the union U of all the $Z_1(i)$ with $i \in I$ would have density at most $\tau/2$ in G' and thus density at most $3\tau/4$ in G. But $zn/|B^*| - \theta^{1/10}n \leq |U| \leq zn/|B^*|$ by (11). Thus by adding at most $\theta^{1/10}n \ll \tau|U|$ vertices to U if necessary we would obtain a set A that contradicts condition (i) of Lemma 13.

Given $i' \in I$, we call an index $i \in I$ *useful for* i' if $d_{G'}(Z_1(i), Z_1(i')) > 0$. Let $I' \subseteq I$ be the set of all those indices $i' \in I$ for which at least $\tau|I|/8$ other indices $i \in I$ are useful. Thus

$$|I'| \geq \tau|I|/8. \tag{12}$$

Note that for every pair $i, i' \in I$ there exists an index $j' = j'(i, i')$ with $1 \le j' < \ell$ and such that $d_{G'}(Z_{j'}(i), Z_1(i')) = 0$. (Otherwise we could take i', i for i_1, i_2.) So in the graph G' every vertex in $Z_1(i')$ has at most $(\ell - 2 + \xi/(1-\theta))L''$ neighbours in $Z_1(i) \cup \cdots \cup Z_\ell(i)$. Clearly, $2 \le j' < \ell$ if i is useful for i'.

Given $i' \in I$, call another index $i \in I$ *typical for* i' if $d_{G'}(Z_j(i), Z_1(i')) > 0$ for all $j \le \ell$ with $j \ne j'$. Thus if i is not typical for i' then in the graph G' every vertex in $Z_1(i')$ has at most $(\ell-2)L''$ neighbours in $Z_1(i) \cup \cdots \cup Z_\ell(i)$. Given $i' \in I'$, we will now show that at least half of the $\ge \tau|I|/8$ indices i which are useful for i' are also typical. Indeed, suppose not. Consider any vertex $v \in Z_1(i')$ and look at its degree in G'. We have that

$$\begin{aligned} d_{G'}(v) &\le |I|L''(1-\tau/16)(\ell - 2 + \xi/(1-\theta)) + \frac{\tau}{16}|I|L''(\ell-2) + \theta^{1/5}n \\ &\overset{(11)}{\le} (\ell - 2 + \xi)k''L'' - \tau\xi|I|L''/16 + 2\theta^{1/5}n \le \delta(G) - \tau^2 n < \delta(G'), \end{aligned}$$

a contradiction. (Indeed, to see the first inequality use that the error bound $\theta^{1/5}n$ on the right hand side is a bound on the number of all those neighbours of v which lie in V_0 or in sets $Z_j(i)$ with $i \notin I$ (c.f. (10) and (11)). Again, we have room to spare here. To check the third inequality use that (9) implies $n - |V_0| = (\ell - 1 + \xi/(1-\theta))L''k'' \ge (\ell - 1 + \xi)L''k''$. For the last inequality use that the Regularity lemma (Lemma 9) implies $\delta(G') \ge \delta(G) - 2dn$.) This shows that for every $i' \in I'$ at least $\tau|I|/16$ indices $i \in I$ are both useful and typical for i'.

Consider all the triples i, i', j' such that $i \in I$, $i' \in I'$ and such that i is both useful and typical for i' and where $j' = j'(i, i')$ is as defined after (12). It is easy to see that the number of such triples is at least $\tau|I||I'|/16$. Thus there must be one pair i, j' which occurs for at least $\tau|I'|/(16\ell)$ indices $i' \in I'$. Let I'' denote the set of all these indices i'. So crudely

$$|I''| \overset{(12)}{\ge} \tau^3|I| \overset{(11)}{\ge} \tau^4 k''. \tag{13}$$

Note that for each $i' \in I''$ there exists a j'' such that $2 \le j'' \le \ell$ and $d_{G'}(Z_{j'}(i), Z_{j''}(i')) = 0$. (Otherwise we could take i, i', j' for i_3, i_4, j^* since i is both useful and typical for i'.) So in the graph G' every vertex in $Z_{j'}(i)$ has at most $(\ell-2)L''$ neighbours in $Z_1(i') \cup \cdots \cup Z_\ell(i')$.

Furthermore, for each $i'' \in I \setminus (I'' \cup \{i\})$ there exists a j''' such that $1 \le j''' < \ell$ and $d_{G'}(Z_{j'}(i), Z_{j'''}(i'')) = 0$. (Otherwise we could take i, i', i'', j' for $i_5, i_6, i_7, j^\diamond$.) Thus for each $i'' \in I \setminus (I'' \cup \{i\})$ we can still say that in G' a vertex $v \in Z_{j'}(i)$ has at most $(\ell - 2 + \xi/(1-\theta))L''$ neighbours in $Z_1(i'') \cup \cdots \cup Z_\ell(i'')$. Note that v sends at most $\theta^{1/5}n$ edges to V_0 and to sets Z_{i^*} with $i^* \notin I$ by (10) and (11). Together the above observations show that

$$\begin{aligned} d_{G'}(v) &\le (|I| - |I''|)(\ell - 2 + \xi/(1-\theta))L'' + |I''|(\ell-2)L'' + \theta^{1/5}n \\ &\le k''(\ell - 2 + \xi)L'' - \xi|I''|L''/(1-\theta) + 2\theta^{1/5}n \overset{(13)}{\le} \delta(G) - \tau^5 n < \delta(G'), \end{aligned}$$

a contradiction.

Thus we have shown that we can incorporate the first exceptional vertex x by removing at most three copies of B^* which are as in $(*)$. Recall that this ensures that the subsets thus obtained from the $Z_j(i)$ satisfy conditions (a) and (c). Next we proceed similarly with all other vertices in V_0. However, in order to ensure that in the end condition (b) is satisfied too, we need to be careful that we do not remove to many vertices from a single set $Z_j(i)$. So if the size of some set $Z_j(i)$ becomes critical after we dealt with some vertex in V_0, then we exclude *all* the sets $Z_1(i), \dots, Z_\ell(i)$ from consideration when dealing with

the remaining vertices in V_0. The definition of the critical threshold in (b) implies that we exclude at most

$$z|V_0|/(\theta^{1/7}L'') \stackrel{(10)}{\leq} 4z\sqrt{\theta}n/(\theta^{1/7}L'') \ll \theta^{1/3}k''$$

indices i in this way. It is easy to check that this will not affect any of the above calculations significantly. This completes the proof of the claim.

Recall that the sets $Z_j(i)$ were obtained by splitting the clusters belonging to the copies $B_1, \dots, B_{k'}$ of B^* in $\mathcal{B}'$. By taking out the copies of B^* chosen in the above process we modified these clusters. For all $t \leq k'$ and all $j \leq \ell$ let $X_j(t)$ denote the union of all the modified clusters belonging to the jth vertex class of B_t. Thus $X_j(t) = Y_j((t-1)z_1z + 1) \cup \dots \cup Y_j(tz_1z)$. Moreover, (9) and (a)–(c) imply that

- (a$'$) all the bipartite graphs $(X_j(t), X_{j'}(t))_{G'}$ are (ε', d')-superregular whenever $j \neq j'$,
- (b$'$) $(1-\theta^{1/8})zL' \leq |X_j(t)| \leq (1-\theta)zL'$ for all $j < \ell$ and $(1-\theta^{1/8})z_1L' \leq |X_\ell(t)| \leq z_1L'$,
- (c$'$) $|X_1(t)| = \dots = |X_{\ell-1}(t)| \leq (1-\theta)|X_\ell(t)|/\xi$.

5.3. **Making the blow-ups of the $B \in \mathcal{B}'$ divisible by $|B^*|$.** Given a subgraph $S \subseteq R$, we denote by $V_G(S) \subseteq V(G)$ the blow-up of $V(S)$. Thus $V_G(S)$ is the union of all the clusters which are vertices of S. In particular, $V_G(B_i) = X_1(i) \cup \dots \cup X_\ell(i)$. If $|V_G(B_i)|$ was divisible by $|B^*|$ for each $B_i \in \mathcal{B}'$, then $\mathcal{B}'$ would correspond to a blown-up B^* cover as required in the lemma. As already described at the end of Section 5.1, we will achieve this by taking out a bounded number of further copies of B^* from G. For this, we define an auxiliary graph F whose vertices are the elements of $\mathcal{B}'$ and in which $B_i, B_j \in \mathcal{B}'$ are adjacent if the reduced graph R contains a copy of K_ℓ with one vertex in B_i and $\ell - 1$ vertices in B_j or vice versa.

To motivate the definition of F, let us first consider the case when F is connected. If $B_i, B_j \in \mathcal{B}'$ are adjacent in F then G contains a copy of B^* with one vertex in $V_G(B_i)$ and all the other vertices in $V_G(B_j)$ or vice versa. In fact, we can even find $|B^*|-1$ disjoint such copies of B^* in G. Taking out a suitable number of such copies (at most $|B^*|-1$), we can achieve that the size of the subset of $V_G(B_i)$ obtained in this way is divisible by $|B^*|$. Thus we can 'shift the remainders mod $|B^*|$' along a spanning tree of F to achieve that $|V_G(B)|$ is divisible by $|B^*|$ for each $B \in \mathcal{B}^*$. (To see this, use that $\sum_{B\in\mathcal{B}'}|V_G(B)|$ is divisible by $|B^*|$ since $|G|$ is divisible by $|B^*|$.)

Let us next show that in the case when $\ell = 2$ the graph F is always connected. If $\ell = 2$, then $B_i, B_j \in \mathcal{B}'$ are joined in F if and only R contains an edge between B_i and B_j. Now suppose that F is not connected and let C be any component of F. Let $A \subseteq V(G)$ denote the union of all those clusters which belong to some $B_i \in C$. Then in the current subgraph of G' there are no edges emanating from A. As we have taken out at most $3|B^*||V_0| \leq 12|B^*|\sqrt{\theta}n$ vertices in Section 5.2 this implies that $d_{G'}(A, V(G') \setminus A) \leq \theta^{1/3}$. Since $d_G(x) \leq d_{G'}(x) + 2dn$ for any $x \in G$ we have $d_G(A, V(G) \setminus A) \leq \theta^{1/3} + 4d \ll \tau$, a contradiction to condition (ii) of Lemma 13.

Thus in what follows we may assume that $\ell \geq 3$ and that F is not connected. Let $\mathcal{C}$ denote the set of all components of F. Given a component C of F, we denote by $V_R(C) \subseteq V(R)$ the set of all those clusters which belong to some $B \in \mathcal{B}'$ with $B \in C$. Let $V_G(C) \subseteq V(G)$ denote the union of all clusters in $V_R(C)$. We first show that we can take out a bounded number of copies of B^* from G in order to make $|V_G(C)|$ divisible by $|B^*|$ for each $C \in \mathcal{C}$. After that, we can 'shift the remainders mod $|B^*|$' within each component

$C \in \mathcal{C}$ along a spanning tree as indicated above to make $|V_G(B)|$ divisible by $|B^*|$ for each $B \in \mathcal{B}'$. For our argument, we will need the following claim.

Claim 1. *Let $C_1, C_2 \in \mathcal{C}$ be distinct and let $a \in V_R(C_2)$. Then*

$$|N_R(a) \cap V_R(C_1)| < \frac{\ell - 2}{\ell - 1}|V_R(C_1)|.$$

Suppose not. Then there is some $B \in \mathcal{B}'$ such that $B \in C_1$ and such that

$$|N_R(a) \cap B| \geq \frac{\ell - 2}{\ell - 1}|B| > \frac{\ell - 2}{\ell - 1 + \xi}|B| = (\ell - 2)z.$$

This implies that a has a neighbour in at least $\ell - 1$ vertex classes of B. Thus R contains a copy of K_ℓ which consists of a together with $\ell - 1$ of its neighbours in B. But by definition of the auxiliary graph F, this means that B is adjacent in F to the copy $B_i \in \mathcal{B}'$ that contains a, i.e. B and B_i lie in the same component of F, a contradiction. This completes the proof of Claim 1.

Claim 2. *There exist a component $C' \in \mathcal{C}$, a copy K of K_ℓ in R and a vertex $a_0 \in V(R) \setminus (V(K) \cup V_R(C'))$ such that K meets $V_R(C')$ in exactly one vertex and such that a_0 is joined to all the remaining vertices in K.*

As $\delta(R) > 1/2$, there exists an edge $a_1a_2 \in R$ which joins the vertex sets corresponding to two different components of F, i.e. there are distinct $C_1, C_2 \in \mathcal{C}$ such that $a_1 \in V_R(C_1)$ and $a_2 \in V_R(C_2)$. Note that (8) implies that

$$\delta(R) > \frac{\ell - 2}{\ell - 1}|R|. \tag{14}$$

Thus the number of common neighbours of a_1 and a_2 in R is greater than $\frac{\ell-3}{\ell-1}|R|$. To prove the claim, we will now distinguish two cases.

Case 1. *More than $\frac{\ell-3}{\ell-1}|V(R) \setminus V_R(C_1)|$ common neighbours of a_1 and a_2 lie outside $V_R(C_1)$.*

Let a_3 be a common neighbour of a_1 and a_2 outside $V_R(C_1)$. Claim 1 and (14) together imply that the number of common neighbours of a_1, a_2 and a_3 outside $V_R(C_1)$ is more than

$$\frac{\ell - 4}{\ell - 1}|V(R) \setminus V_R(C_1)|.$$

Choose such a common neighbour a_4. Continuing in this way, we can obtain distinct vertices $a_2, \dots, a_\ell$ outside $V_R(C_1)$ which together with a_1 form a copy K of K_ℓ in R. As before, Claim 1 and (14) together imply that the number of common neighbours of $a_2, \dots, a_\ell$ outside $V_R(C_1)$ is nonzero. Let a_0 be such a common neighbour. Then Claim 2 holds with $C' := C_1$, K and a_0. Thus we may now consider

Case 2. *More than $\frac{\ell-3}{\ell-1}|V_R(C_1)|$ common neighbours of a_1 and a_2 lie in $V_R(C_1)$.*

In this case we proceed similarly as in Case 1. However, this time we choose $a_0, a_3, \dots, a_\ell$ inside $V_R(C_1)$. Indeed, this can be done since Claim 1 and (14) together imply that each vertex in $V_R(C_1)$ has more than $\frac{\ell-2}{\ell-1}|V_R(C_1)|$ neighbours in $V_R(C_1)$. Then Claim 2 holds with $C' := C_2$.

Claim 3. *We can make $|V_G(B)|$ divisible by $|B^*|$ for all $B \in \mathcal{B}'$ by taking out at most $|\mathcal{B}'||B^*|$ disjoint copies of B^* from G.*

We first take out some copies of B^* from G to achieve that $|V_G(C)|$ is divisible by $|B^*|$ for each $C \in \mathcal{C}$. To do this we proceed as follows. We apply Claim 2 to find a component

$C_1 \in \mathcal{C}$, a copy K of K_ℓ in R and a vertex $a_0 \in V(R)\setminus(V(K)\cup V_R(C_1))$ such that K meets $V_R(C_1)$ in exactly one vertex, a_1 say, and such that a_0 is joined to all vertices in $K - a_1$. Thus G contains a copy B' of B^* which has exactly one vertex $x \in V_G(C_1)$ and whose other vertices lie in clusters belonging to $V(K - a_1) \cup \{a_0\}$. (Indeed, we can choose the vertices of B' lying in the same vertex class as x in the cluster a_0 and the vertices lying in other vertex classes in the clusters belonging to $K - a_1$.) In fact, G contains $|B^*| - 1$ (say) disjoint such copies of B^*. Now suppose that $|V_G(C_1)| \equiv j \mod |B^*|$. Then we take out j disjoint such copies of B^* from G to achieve that $|V_G(C_1)|$ is divisible by $|B^*|$. Next we consider the graphs $F_1 := F - V(C_1)$ and $R_1 := R - V_R(C_1)$ instead of F and R. Claim 1 and (14) together imply that $\delta(R_1) > \frac{\ell-2}{\ell-1}|R_1|$. Now suppose that $|\mathcal{C}| \geq 3$. Then similarly as in the proof of Claim 2 one can find a component $C_2 \in \mathcal{C} \setminus \{C_1\}$, a copy K' of K_ℓ in R_1 and a vertex $a'_0 \in V(R_1) \setminus (V(K') \cup V_R(C_2))$ such that K' meets $V_R(C_2)$ in exactly one vertex, a_2 say, and such that a'_0 is joined to all vertices in $K - a_2$. As before, we take out at most $|B^*| - 1$ copies of B^* from G to achieve that $|V_G(C_2)|$ is divisible by $|B^*|$. As $|G|$ was divisible by $|B^*|$, we can continue in this fashion to achieve that $|V_G(C)|$ is divisible by $|B^*|$ for all components $C \in \mathcal{C}$. In this process, we have to take out at most $(|\mathcal{C}| - 1)(|B^*| - 1)$ copies of B^* from G. Now we consider each component $C \in \mathcal{C}$ separately. By proceeding as in the connected case for each C and taking out at most $(|C| - 1)(|B^*| - 1)$ further copies of B^* from G in each case, we can make $|V_G(B)|$ divisible by $|B^*|$ for each $B \in \mathcal{B}'$. Hence, in total, we have taken out at most $(|\mathcal{C}| - 1)(|B^*| - 1) + (|\mathcal{B}'| - |\mathcal{C}|)(|B^*| - 1) \leq |\mathcal{B}'||B^*|$ copies of B^* from G.

$\mathcal{B}'$ now corresponds to a blown-up B^*-cover as desired in the lemma, except that it has $k' \leq k_1$ elements. (Recall that k_1 was defined in (6).) But by considering random partitions, it is easy to see that one can split these elements to obtain a blown-up B^*-cover as required. The B^*-packing $\mathcal{B}^*$ in Lemma 13 consists of all the copies of B^* taken out during the proof. Thus $|\mathcal{B}^*| \leq 3|V_0| + |\mathcal{B}'||B^*| \leq 12\sqrt{\theta}n + |\mathcal{B}'||B^*| \leq \theta^{1/3}n$, as desired.

6. Packings in complete ℓ-partite graphs

In this section, we prove several results which together imply Lemma 12. However, almost all of the results of this section are also used directly in Section 7.

Clearly, a complete ℓ-partite graph has a perfect H-packing if all its vertex classes have equal size which is divisible by $|H|$. Together the following two lemmas show that if $\mathrm{hcf}(H) = 1$ then we still have a perfect H-packing if the sizes of the vertex classes are permitted to deviate slightly. (By Proposition 6 this is false if $\chi(H) \geq 3$ and $\mathrm{hcf}(H) \neq 1$ or if $\chi(H) = 2$ and $\mathrm{hcf}_\chi(H) > 2$.) In Lemma 15 we first consider the case when $\mathrm{hcf}_\chi(H) = 1$. In Lemma 16 we then deal with the remaining case (i.e. when H is bipartite, $\mathrm{hcf}(H) = 1$ but $\mathrm{hcf}_\chi(H) = 2$).

Lemma 15. *Suppose that H is a graph of chromatic number $\ell \geq 2$ such that $\mathrm{hcf}_\chi(H) = 1$. Let B^* be the bottlegraph assigned to H. Let $D' \gg |H|$ be an integer divisible by $|H|$. Let a be an integer such that $|a| \leq |B^*|$. Given $1 \leq i_1 < i_2 \leq \ell$, let G be a complete ℓ-partite graph with vertex classes $U_1, \dots, U_\ell$ such that $|U_{i_1}| = D' + a$, $|U_{i_2}| = D' - a$ and $|U_r| = D'$ for all $r \neq i_1, i_2$. Then G contains a perfect H-packing.*

Proof. Let q denote the number of optimal colourings of H. First note that by taking out at most $q\ell!$ disjoint copies of H from G we may assume that D' is divisible by $q(\ell-1)!|H|$. Consider the complete ℓ-partite graph G' whose vertex classes $U'_1, \dots, U'_\ell$ all have size D'. Thus $|G| = |G'|$. We will think of G and G' as two graphs on the same vertex set whose vertex classes are roughly identical. Recall that G' has a perfect H-packing.

Our aim is to choose a suitable such H-packing $\mathcal{H}'$ in G' and to show that it can be modified into a perfect H-packing of G. To choose $\mathcal{H}'$, we will consider all optimal colourings of H. Let $c^1, \dots, c^q$ be all these colourings. Let $x_1^j \le x_2^j \le \dots \le x_\ell^j$ denote the sizes of the colour classes of c^j. Put

$$k := \frac{D'}{q(\ell-1)!|H|}.$$

Let S_ℓ denote the set of all permutations of $\{1, \dots, \ell\}$. Given $j \le q$, let $\mathcal{H}'_j$ be an H-packing in G' which, for all $s \in S_\ell$, contains precisely k copies of H which in the colouring c^j have their $s(i)$th colour class in U'_i (for all $i = 1, \dots, \ell$). Thus $\mathcal{H}'_j$ consists of $\ell!k$ copies of H and covers precisely $k(\ell-1)!|H|$ vertices in each vertex class U'_i of G'. Moreover, we choose all the $\mathcal{H}'_j$ to be disjoint from each other. Thus the union $\mathcal{H}'$ of $\mathcal{H}'_1, \dots, \mathcal{H}'_q$ is a perfect H-packing in G'.

We will now show that $\mathcal{H}'$ can be modified into a perfect H-packing of G. Roughly, the reason why this can be done is the following. Clearly, we may assume that $a > 0$. So $\mathcal{H}'$ has less than $|U_{i_1}|$ vertices in its i_1th vertex class and more than $|U_{i_2}|$ vertices in its i_2th vertex class. We will modify $\mathcal{H}'$ slightly by interchanging some vertex classes in some copies of H in $\mathcal{H}'$. As $\mathrm{hcf}_\chi(H) = 1$ this can be done in such a way that the H-packing obtained from $\mathcal{H}'$ covers one vertex more in its i_1th vertex class than $\mathcal{H}'$ and one vertex less in its i_2th vertex class. Continuing in this fashion we obtain an H-packing which covers the correct number of vertices in each vertex class.

For all $j \le q$ and $r < \ell$ put $d_r^j := x_{r+1}^j - x_r^j$. Since $\mathrm{hcf}_\chi(H) = 1$, we can find $b_r^j \in \mathbb{Z}$ such that

$$1 = \sum_{j=1}^{q} \sum_{r=1}^{\ell-1} b_r^j d_r^j.$$

(Here we take $b_r^j := 0$ if $d_r^j = 0$.) In order to modify the H-packing $\mathcal{H}'$ we proceed as follows. For all $j \le q$ and $r < \ell$ we consider b_r^j. If $b_r^j \ge 0$ we choose b_r^j of the copies of H in $\mathcal{H}'_j \subseteq \mathcal{H}'$ which in the colouring c^j have their rth vertex class in U'_{i_1} and their $(r+1)$th vertex class in U'_{i_2}. We change each of these copies of H such that they now have their rth vertex class in U'_{i_2} and its $(r+1)$th vertex class in U'_{i_1}. All the other vertices remain unchanged. Note the number of vertices in the i_1th vertex class covered by this new H-packing increases by $b_r^j d_r^j$ whereas the number of covered vertices in the i_2th vertex class decreases by $b_r^j d_r^j$.

If $b_r^j < 0$ we choose $|b_r^j|$ of the copies of H in $\mathcal{H}'_j$ which in the colouring c^j have their rth vertex class in U'_{i_2} and their $(r+1)$th vertex class in U'_{i_1}. This time, we change each of these copies of H such that they now have their rth vertex class in U'_{i_1} and its $(r+1)$th vertex class in U'_{i_2}. Note that all these copies of H will automatically be distinct for different pairs j, r.

Let $\mathcal{H}^*$ denote the modified H-packing obtained by proceeding as described above a times (where all the copies of H which we change are chosen to be distinct). We have to check that $\mathcal{H}^*$ is a perfect H-packing of G. For all $i \le \ell$ let n_i denote the number of vertices in the ith vertex class covered by $\mathcal{H}^*$. Thus $n_i = D'$ whenever $i \ne i_1, i_2$. We have to check that $n_{i_1} = D' + a$ and $n_{i_2} = D' - a$. But

$$n_{i_1} = D' + a \sum_{j=1}^{q} \sum_{r=1}^{\ell-1} b_r^j d_r^j = D' + a$$

and

$$n_{i_2} = D' - a \sum_{j=1}^{q} \sum_{r=1}^{\ell-1} b_r^j d_r^j = D' - a,$$

as required. □

Lemma 16. *Suppose that H is a bipartite graph such that $\mathrm{hcf}_c(H) = 1$ and $\mathrm{hcf}_\chi(H) = 2$. Let B^* be the bottlegraph assigned to H. Let $D' \gg |H|$ be an integer divisible by $|H|$. Let a be an integer such that $|a| \le |B^*|$. Let G be a complete bipartite graph with vertex classes U_1 and U_2 such that $|U_1| = D' + a$ and $|U_2| = D' - a$. Then G contains a perfect H-packing.*

Proof. Clearly, we may assume that $a > 0$. Our first aim is to take out a small number of disjoint copies of H from G to obtain sets $U_i' \subseteq U_i$ with $|U_1'| = |U_2'|$. To do this, we will use the fact that $\mathrm{hcf}_\chi(H) = 2$. So let $c^1, \dots, c^q$ be all the optimal colourings of H. Let $x_1^j \le x_2^j$ denote the sizes of the colour classes of c^j. Since $\mathrm{hcf}_\chi(H) = 2$, we can find $b^j \in \mathbb{Z}$ such that $2 = \sum_{j=1}^q b^j(x_2^j - x_1^j)$. (Here we take $b^j := 0$ if $x_2^j = x_1^j$.) For each $j = 1, \dots, q$ in turn we take out $a|b^j|$ copies of H from G. If $b^j \ge 0$ each of these $a|b^j|$ copies will meet U_1 in x_2^j vertices and U_2 in x_1^j vertices. If $b^j < 0$ then each of these copies will have x_1^j vertices in U_1 and x_2^j vertices in U_2. We choose all these copies of H to be disjoint. It is easy to check that the subsets U_1' and U_2' obtained from U_1 and U_2 in this way have the same size, u' say. Note that $|H|$ divides $2u'$ since $|H|$ divides $|U_1|+|U_2|$. Also observe that if $|H|$ even divides u', then $G[U_1' \cup U_2']$ (and thus also G itself) has a perfect H-packing. So we may assume that $|H|$ does not divide u' but that it does divide $2u'$. Hence $|H|$ is even and $u' = |H|k/2$ where k is an odd integer.

We will now use the fact that $\mathrm{hcf}_c(H) = 1$ to show that we can take out further copies of H from G to achieve that the subsets U_1'' and U_2'' obtained in this way have the same size and that this size is divisible by $|H|$. As $\mathrm{hcf}_c(H) = 1$ there exists a component C of H such that $|C|$ is odd. Using the fact that $|H|$ is even and thus $|H - C|$ is odd it is easy to see that there exists a 2-colouring of H whose colour classes both have odd size and another 2-colouring whose colour classes both have even size. We may assume that c^1 and c^2 are such colourings, i.e. that both x_1^1 and x_2^1 are odd and both x_1^2 and x_2^2 are even. Let $k_1 := |H|/2 - x_1^2$ and $k_2 := |H|/2 - x_1^1$. Take out k_1 copies of H with x_1^1 vertices in U_1' and x_2^1 vertices in U_2'. Then take out k_2 copies of H with x_2^2 vertices in U_1' and x_1^2 vertices in U_2'. Let U_1'' and U_2'' denote the subsets obtained from U_1' and U_2' in this way. It is easy to check that

$$|U_1''| = |U_2''| = u' - \frac{|H|}{2}(|H| - x_1^1 - x_1^2) = \frac{|H|}{2}(k - |H| + x_1^1 + x_1^2).$$

But $k - |H| + x_1^1 + x_1^2$ is even and so $|U_1''|$ is divisible by $|H|$, as desired. □

The next lemma is an analogue of Lemma 16 for perfect H-packings in graphs which are the disjoint union of two cliques. It will be needed in the proof of Lemma 23.

Lemma 17. *Suppose that H is a bipartite graph such that $\mathrm{hcf}_c(H) = 1$. Let B^* be the bottlegraph assigned to H. Let $D' \gg |H|$ be an integer divisible by $|H|$. Let a be an integer such that $|a| \le |B^*|$. Let G be the disjoint union of two cliques of order $D' + a$ and $D' - a$ respectively. Then G contains a perfect H-packing.*

Proof. Clearly, we may assume that $a > 0$. Let G_1 be the clique of order $D' + a$ and let G_2 be the clique of order $D' - a$. Our aim is to take out disjoint copies of H from G in order to obtain subcliques $G'_i \subseteq G_i$ such that $|G'_i|$ is divisible by $|H|$ for both $i = 1, 2$. (Then each G'_i (and thus also G itself) has a perfect H-packing.)

Let $C_1 \leq \dots \leq C_s$ denote the components of H. Since $\mathrm{hcf}_c(H) = 1$, we can find $b^j \in \mathbb{Z}$ such that $1 = \sum_{j=1}^{s} b^j |C_j|$. For each $j = 1, \dots, s$ in turn we take out $a|b^j|$ copies of H from G. If $b^j \geq 0$ each of these $a|b^j|$ copies will meet G_1 in C_j and G_2 in $H - C_j$. If $b^j < 0$ then each of these copies will meet G_1 in $H - C_j$ and G_2 in C_j. We choose all these copies of H to be disjoint. Let G'_1 and G'_2 denote the subcliques obtained from G_1 and G_2 in this way. Then

$$|G'_1| = D' + a - a \sum_{j:\, b^j \geq 0} b_j |C_j| + a \sum_{j:\, b^j < 0} b^j (|H| - |C_j|) = D' + a \sum_{j:\, b^j < 0} b^j |H|.$$

Thus $|H|$ divides $|G'_1|$. Similarly one can show that $|H|$ divides $|G'_2|$. □

The following lemma states that if G is a complete ℓ-partite graph which is very close to being bottle-shaped then G contains a perfect H-packing as long as the ratio of the smallest to the largest vertex class a bit larger than in the bottlegraph B^* of H. (The terms involving D' in (i) and (ii) ensure that the latter condition holds.)

Lemma 18. *Suppose that H is a graph of chromatic number $\ell \geq 2$ such that $\mathrm{hcf}(H) = 1$. Let B^* be the bottlegraph assigned to H. Let z and z_1 be as defined in (2). Let $D' \gg |H|$ be an integer divisible by $|B^*|$. Let G be a complete ℓ-partite graph with vertex classes $U_1, \dots, U_\ell$ whose order $n \gg D'$ is divisible by $|B^*|$. Let $u_i := |U_i|$ for all i. Suppose that*

(i) $|\,(u_i - D') - z(n - \ell D')/|B^*|\,| \leq |B^*|$ *for all $i < \ell$ and*
(ii) $|\,(u_\ell - D') - z_1(n - \ell D')/|B^*|\,| \leq |B^*|$.

Then one can take out $\ell D'/|H|$ disjoint copies of H from G to obtain a subgraph $G^* \subseteq G$ such that, writing $n^* := |G^*| = n - \ell D'$ and $u_i^* := |U_i \cap V(G^*)|$, we have $u_i^* = zn^*/|B^*|$ for all $i < \ell$ and $u_\ell^* = z_1 n^*/|B^*|$. So in particular, G^* contains a perfect B^*-packing and thus G contains a perfect H-packing.

Proof. Let us first consider the case when $\mathrm{hcf}_\chi(H) = 1$. Put $a_i := (u_i - D') - z(n - \ell D')/|B^*|$ for all $i < \ell$ and $a_\ell := (u_\ell - D') - z_1(n - \ell D')/|B^*|$. Thus $\sum_{i=1}^{\ell} a_i = 0$ and $|a_i| \leq |B^*|$ for all $i \leq \ell$. Consider the complete ℓ-partite graph G' whose ith vertex class has size $D' + a_i$. By repeated applications of Lemma 15 one can show that this graph has a perfect H-packing $\mathcal{H}$. View G' as a subgraph of G such that the ith vertex class of G' lies in U_i. Then the subgraph G^* obtained from G by removing all the copies of H in $\mathcal{H}$ (and thus deleting precisely the vertices in $V(G') \subseteq V(G)$) is as required in the lemma. In the remaining case when $\mathrm{hcf}_\chi(H) \neq 1$ (and thus $\chi(H) = 2$, $\mathrm{hcf}_c(H) = 1$ and $\mathrm{hcf}_\chi(H) = 2$) we proceed similarly except that we now apply Lemma 16 instead of Lemma 15. □

The next lemma shows that we can achieve the conditions in the setup of Lemma 18 when larger deviations from the bottle-shape are allowed.

Lemma 19. *Let H, B^*, z, z_1, G, U_i, u_i and D' be defined as in the previous lemma, except that we now no longer assume that $\mathrm{hcf}(H) = 1$ and that G satisfies (i) and (ii) and we only require $D' \geq 0$ to be any integer divisible by $|B^*|$. Suppose that $\chi_{cr}(H) < \ell$ and $a_i := z(n - \ell D')/|B^*| - (u_i - D') \geq 0$ where $a_i \leq n/(\ell^3 |B^*|^2)$ for all $i < \ell$. Then one can take out at most $\ell^2 \sum_{i=1}^{\ell-1} a_i$ disjoint copies of B^* from G to obtain a subgraph $G^* \subseteq G$*

such that, writing $n^* := |G^*|$ and $u_i^* := |U_i \cap V(G^*)|$, conditions (i) and (ii) of Lemma 18 hold with n replaced by n^* and with u_i replaced by u_i^*.

Proof. First note that we only need to consider the case when $D' = 0$. Indeed, to reduce the general case, suppose that Lemma 19 holds if $D' = 0$. Now instead of G we consider the graph G' obtained from G by removing D' vertices from each vertex class. Apply Lemma 19 to G' to obtain a graph $G^* \subseteq G'$. Let $U_1^*, \dots, U_\ell^*$ denote the vertex classes of G^*. Then the vertex classes obtained from the U_i^* by adding the D' vertices span a subgraph of G as desired in the lemma. We may also assume that $u_1 \leq \dots \leq u_{\ell-1}$. Let $k := n/|B^*|$ and let ξ be as defined in (2). Note that

$$u_i = kz - a_i \text{ for } i < \ell \quad \text{and} \quad u_\ell = \xi z k + \sum_{i=1}^{\ell-1} a_i. \tag{15}$$

We now take out disjoint copies of B^* from G in order to achieve that the subsets of $U_1, \dots, U_{\ell-1}$ thus obtained have almost the same size. More precisely, we proceed as follows. For every $i = 1, \dots, \ell-2$ let $r_i := \lfloor (u_{\ell-1} - u_i)/(z - z_1) \rfloor$. Put

$$r := \sum_{i=1}^{\ell-2} r_i \leq \frac{(\ell-2)(kz - a_{\ell-1}) - \sum_{i=1}^{\ell-2}(kz - a_i)}{z - z_1} = \frac{\sum_{i=1}^{\ell-2} a_i - (\ell-2)a_{\ell-1}}{z - z_1}.$$

For every $i = 1, \dots, \ell-2$ in turn remove r_i copies of B^* from G, each having z_1 vertices in U_i and z vertices in every other set U_j. Then the subsets U_i' obtained from the U_i in this way satisfy $0 \leq |U_{\ell-1}'| - |U_i'| < z - z_1$ for all $i = 1, \dots, \ell-2$ and

$$|U_\ell'| - \xi |U_{\ell-1}'| = u_\ell - \xi u_{\ell-1} - r(z - z_1) \overset{(15)}{\geq} (\ell - 1 + \xi) a_{\ell-1} \geq 0.$$

Next we will take out further copies of B^* from G in order to achieve that the size of the ℓth vertex class is about ξ-times as large as the size of any other vertex class. In each step we remove $\ell-1$ copies of B^*, for every $i = 1, \dots, \ell-1$ one copy having z_1 vertices in the ith vertex class and z vertices in each other class. A straightforward calculation shows that after

$$\left\lfloor \frac{|U_\ell'| - \xi |U_{\ell-1}'|}{(\ell-1)z - (\ell-2)z_1 - \xi z_1} \right\rfloor$$

steps the subsets U_i^* obtained from the U_i' in this way span a subgraph as required in the lemma. □

Proof of Lemma 12. Let D' be an integer as in Lemma 18. Consider the graph F given in Lemma 12. By taking out at most $\ell - 2$ disjoint copies of H from F if necessary, we may assume that $|F|$ is divisible by $|B^*|$. It is easy to check that F satisfies the conditions in Lemma 19 if $|F| \gg D'$. Thus Lemmas 19 and 18 together imply Lemma 12. □

7. Proof of the extremal cases

In most of the extremal cases, we know that G contains several large almost independent sets $A_1, \dots, A_q$ where $1 \leq q < \ell$. In the preliminary Lemma 21 we show that we can modify the A_i slightly to obtain sets $A_1^*, \dots, A_q^*$ which together with $V(G) \setminus \bigcup_{i=1}^q A_i^*$ induce an almost complete $(q+1)$-partite graph.

In the proof of Lemma 21 below we need the following observation.

Lemma 20. *Let i be a positive integer and let G be a graph of order n whose average degree satisfies $d := d(G) \geq 2i$. Then G contains at least*

$$\frac{dn}{4(i+1)\Delta(G)}$$

disjoint i-stars.

Proof. Let $k := \lceil dn/(4(i+1)\Delta(G))\rceil$. We take out the disjoint i-stars greedily. So in each step we delete $i+1$ vertices and thus at most $(i+1)\Delta(G)$ edges. So after $< k$ steps the remaining subgraph G' of G has at least $e(G) - k(i+1)\Delta(G) \geq dn/4$ edges and thus $d(G') \geq d/2 \geq i$. So G' contains an i-star. This shows that the number of disjoint i-stars we can find greedily is at least k. □

Lemma 21. *Suppose that H is a graph of chromatic number $\ell \geq 2$ such that $\chi_{cr}(H) < \ell$. Let B^* denote the bottlegraph assigned to H. Let ξ, z and z_1 be as defined in (2) and let $0 < \tau \ll \xi, 1-\xi, 1/|B^*|$. Let $|B^*| \ll D' \ll C$ be integers such that D' is divisible by $|B^*|$. Let G be a graph whose order $n \gg C, 1/\tau$ is divisible by $|B^*|$ and whose minimum degree satisfies $\delta(G) \geq (1 - \frac{1}{\chi_{cr}(H)})n + C$. Furthermore, suppose that for some $1 \leq q < \ell$ there are disjoint sets $A_1, \dots, A_q \subseteq V(G)$ which satisfy $|A_i| = (n - 2\ell D')z/|B^*| + 2D'$ and $d(A_i) \leq \tau$. Let $A_{q+1} := V(G) \setminus \bigcup_{i=1}^q A_i$. Then there are sets $A_1^*, \dots, A_{q+1}^*$ which satisfy the following properties:*

- (i) *Let A^* denote the union of $A_1^*, \dots, A_{q+1}^*$ and put $n^* := |A^*|$. Then $G - A^*$ has a perfect H-packing. Moreover $n - n^* \leq \tau^{3/5}n$.*
- (ii) *$|A_i^*| = (n^* - \ell D')z/|B^*| + D'$ and $d(A_i^*) \leq \tau^{2/5}$ for all $i \leq q$.*
- (iii) *For all $i, j \leq q+1$ with $j \neq i$ each vertex in A_i^* has at least $(1 - \tau^{1/5})|A_j^*|$ neighbours in A_j^*.*

Proof. First note that the minimum degree condition on G and (3) imply that the neighbourhood of each vertex $x \in G$ can avoid almost $|A_1| = \cdots = |A_q|$ vertices of G but no more. Given an index $i \leq q+1$, we call a vertex $x \in A_i$ *i-bad* if x has at least $\tau^{1/3}|A_i|$ neighbours in A_i. Since $d(A_i) \leq \tau$ for each $i \leq q$, for such i's the number of i-bad vertices is at most $\tau^{2/3}|A_i|$. Call a vertex $x \in A_i$ *i-useless* if x has at most $(1-\tau^{1/4})|A_j|$ neighbours in A_j for some $j \neq i$. Thus if $i \leq q$ every i-useless vertex is also i-bad. In particular, for each $i \leq q$ there are at most $\tau^{2/3}|A_i|$ vertices which are i-useless. To estimate the number u_{q+1} of $(q+1)$-useless vertices we count the number $e(A_{q+1}, V(G) \setminus A_{q+1})$ of edges emanating from A_{q+1}. We have

$$\begin{aligned} q|A_1|\delta(G) - 2\sum_{i=1}^{q} e(A_i) - 2\binom{q}{2}|A_1|^2 &\leq e(A_{q+1}, V(G) \setminus A_{q+1}) \\ &\leq u_{q+1}[(q-1)|A_1| + (1-\tau^{1/4})|A_1|] + (|A_{q+1}| - u_{q+1})q|A_1| \end{aligned}$$

which implies that the number u_{q+1} of $(q+1)$-useless vertices is at most $\tau^{2/3}|A_{q+1}|$. So in total, at most $\tau^{2/3}n$ vertices of G are i-useless for some $i \leq q+1$.

Given $j \neq i$, we call a vertex $x \in A_i$ *j-exceptional* if x has at most $\tau^{1/3}|A_j|$ neighbours in A_j. Thus every such x is both i-useless and i-bad. It will be important later that the number of vertices which are i-useless for some i is much smaller than the number of neighbours in A_j of a non-j-exceptional vertex. By interchanging i-bad vertices with i-exceptional vertices if necessary, we may assume that for each i for which there exist i-exceptional vertices, we don't have i-bad vertices. Note that after we have interchanged

vertices every non-j-exceptional vertex still has at least $\tau^{1/3}|A_j|/2$ neighbours in A_j. Similarly, every non-i-bad vertex still has at most $2\tau^{1/3}|A_i|$ neighbours in A_i and every non-i-useless vertex still has at least $(1-2\tau^{1/4})|A_j|$ neighbours in A_j for every $j \neq i$.

For each index $i \leq q$ in turn we now proceed as follows in order to take care of the i-exceptional vertices. Let $S_i \subseteq V(G) \setminus A_i$ denote the set of i-exceptional vertices and assume that $s_i := |S_i| > 0$. We will choose a set $\mathcal{S}_i$ of s_i disjoint z-stars in $G[A_i]$ and interchange the star centres with the i-exceptional vertices. To show the existence of such stars, note that $\Delta(A_i) \leq 2\tau^{1/3}|A_i|$ since by our assumption no vertex in A_i is i-bad. Moreover, we can bound the number of edges in $G[A_i]$ by

$$\begin{aligned} e(A_i) &\geq \frac{\delta(G)|A_i| - |G-(A_i \cup S_i)||A_i| - e(A_i, S_i)}{2} \\ &\geq \frac{1}{2}|A_i|\left[\left(1-\frac{1}{\ell-1+\xi}\right)n - \left(n - \frac{n}{\ell-1+\xi} - s_i\right) - 2s_i\tau^{1/3} + \frac{C}{2}\right] \\ &\geq \frac{1}{2}|A_i|\left(C/2 + s_i/2\right). \end{aligned}$$

We only have $C/2$ instead of C in the second line since we have to compensate for the fact that the size of the A_i's is not exactly $n/(\ell-1+\xi)$. Thus $G[A_i]$ has average degree at least $C/2 + s_i/2 \geq 2z$. Lemma 20 now implies that $G[A_i]$ contains at least

$$\frac{(C/2+s_i/2)|A_i|}{8(z+1)\tau^{1/3}|A_i|} \geq s_i$$

disjoint z-stars, as required. We still denote the modified sets by A_i and let $\mathcal{S} := \bigcup_{i=1}^{q} \mathcal{S}_i$.

We now choose a set $\mathcal{B}$ of $|\mathcal{S}|$ disjoint copies of B^* in G, each containing precisely one of the stars in $\mathcal{S}$. Moreover, each such copy will have precisely z vertices in every A_i with $i \leq q$. To see that such copies exist, we will first show that $G[A_{q+1}]$ contains many disjoint copies of B_1^*, where B_1^* denotes the subgraph of B^* obtained by removing q of the large vertex classes. For this, let $n_{q+1} := |A_{q+1}|$. Then

$$\begin{aligned} \frac{\delta(G[A_{q+1}])}{n_{q+1}} &\geq \frac{\delta(G) - |A_1 \cup \dots \cup A_q|}{n} \cdot \frac{n}{n_{q+1}} \\ &\geq \left(1 - \frac{1}{\ell-1+\xi} - \frac{q}{\ell-1+\xi}\right)\frac{\ell-1+\xi}{\ell-q-1+\xi} \\ &= 1 - \frac{1}{\ell-q-1+\xi} \\ &= 1 - \frac{1}{\chi_{cr}(B_1^*)}. \end{aligned} \tag{16}$$

(The fact that $C \gg D'$ enables us to ignore the terms involving the constant D' when estimating n/n_{q+1}.) Since $\tau^{1/5} \ll \chi_{cr}(B_1^*) - (\chi(B_1^*) - 1)$, by repeated applications of the Erdős-Stone theorem (see (1)) we can find $\tau^{1/5}n_{q+1}$ disjoint copies of B_1^* in $G[A_{q+1}]$. Since at most $\tau^{2/3}n_{q+1}$ vertices in A_{q+1} are $(q+1)$-useless, we may assume that all of these copies of B_1^* avoid the $(q+1)$-useless vertices. Moreover, all the stars $S \in \mathcal{S}$ are disjoint and it is easy to see that none of the vertices of such a star S can be i-useless where A_i is the set which originally contained S. The latter implies that each vertex of S is joined to at least $(1-2\tau^{1/4})|A_j|$ vertices in A_j for every $j \neq i$. Thus in particular, each vertex of S is joined to all vertices in almost all of the copies of B_1^* selected above. Altogether, the above shows that we can greedily choose the set $\mathcal{B}$ of $|\mathcal{S}|$ disjoint copies of B^* as follows: for each copy, first choose a star $S \in \mathcal{S}$, then choose a copy of B_1^* selected

above all of whose vertices are joined to all vertices in S. If the centre of S was moved into A_{q+1}, we interchange it with some vertex in the copy of B_1^*. Finally we choose the remaining vertices of B^*. Let A_i' be the subset of A_i which contains all those vertices that do not lie in a copy of B^* in $\mathcal{B}$. Note that

$$|A_i \setminus A_i'| \le |B^*||\mathcal{S}| \le |B^*|\tau^{2/3}n. \tag{17}$$

After this process we have removed all the i-exceptional vertices for all $i \le q$.

The next step is to deal with the useless vertices (and thus also with the $(q+1)$-exceptional vertices). For each such vertex x we will move x into another vertex class or/and we will remove a copy of B^* which contains x. (We do the former if x lies in the set U defined below.) Let U denote the set of all vertices in $A_1' \cup \dots \cup A_q'$ which had at most $(1-\tau^{1/4})|A_{q+1}|$ neighbours in A_{q+1}. So in particular, each $u \in U$ is i-useless where $i \le q$ is the index such that $u \in A_i'$. Thus $|U| \le \tau^{2/3}n$. (Moreover, if $q = \ell - 1$ then U contains all the $(q+1)$-exceptional vertices. If $q < \ell - 1$ then there are no $(q+1)$-exceptional vertices.) Note that each $u \in U \cap A_i'$ must still have at least $\tau^{1/3}|A_i'|$ neighbours in its own class A_i'. Moreover, as in (16) one can show that each $u \in U$ satisfies

$$|N(u) \cap A_{q+1}'| \ge \delta(G) - |A_1 \cup \dots \cup A_q| - |A_{q+1} \setminus A_{q+1}'| \overset{(17)}{\ge} \left(1 - \frac{1}{\chi_{cr}(B_1^*)} - \tau^{3/5}\right)|A_{q+1}'|. \tag{18}$$

Let $A_1'', \dots, A_{q+1}''$ denote the sets obtained from the A_i' by moving all the vertices in U to A_{q+1}'. Then (16) and (18) together with the fact that $\tau^{1/5} \ll \chi_{cr}(B_1^*) - (\chi(B_1^*) - 1)$ imply that

$$\delta(G[A_{q+1}'']) \ge \left(1 - \frac{1}{\chi_{cr}(B_1^*)} - \tau^{1/2}\right)|A_{q+1}''| \ge \left(1 - \frac{1}{\ell - q - 1} + \tau^{1/5}\right)|A_{q+1}''|. \tag{19}$$

(If $q = \ell - 1$ then we will only use the first inequality in (19).) Consider the graph K obtained from the complete $(q+1)$-partite graph with vertex classes $A_1'', \dots, A_{q+1}''$ by making A_{q+1}'' into a clique. Let K'' denote the subgraph of K obtained by deleting D' vertices from each of the first q classes and $(\ell - q)D'$ vertices from A_{q+1}''. An application of Lemmas 19 and 18 shows that by taking out at most $\ell^3|U| + D' \le 2\ell^3\tau^{2/3}n$ disjoint copies of H from K'' one can obtain a subgraph K''' whose vertex classes $A_1''', \dots, A_{q+1}'''$ satisfy $|A_i'''| = z|K'''|/|B^*|$ for all $i \le q$. Moreover, each of these copies of H meets A_{q+1}'' in an $(\ell - q)$-partite graph. Together with (19) and the Erdős-Stone theorem this shows that for each of these copies of H in K'' we can take out a copy of H from G which intersects the $q+1$ vertex classes in exactly the same way and avoids all the useless vertices. We add all these copies of H in G to the set $\mathcal{B}$. Adding the $\ell D'$ vertices set aside (when going from K to K'') to the vertex classes again we thus obtain vertex sets $A_1^\diamond, \dots, A_{q+1}^\diamond$ such that $|A_i^\diamond| = (n^\diamond - \ell D')z/|B^*| + D'$ for all $i \le q$, where $n^\diamond := |A_1^\diamond \cup \dots \cup A_{q+1}^\diamond|$.

By the bound in (17) and the previous paragraph we have removed at most $3\ell^3|B^*|\tau^{2/3}n \ll \tau^{1/3}n$ vertices so far. Thus for all $i \le q+1$ every vertex in $A_i^\diamond$ still has at least $\tau^{1/3}|A_j^\diamond|/3$ neighbours in each other $A_j^\diamond$ with $j \le q$ (since it is non-j-exceptional). Moreover, since we have moved the vertices in U, for all $i \le q$ every vertex in $A_i^\diamond$ still has at least $(1 - 3\tau^{1/4})|A_{q+1}^\diamond|$ neighbours in $A_{q+1}^\diamond$. Also, $d(A_i^\diamond) \le \tau^{2/5}/2$ for all $i \le q$ (note that exchanging i-exceptional vertices with i-bad vertices does not affect the density too much). For all $i \le q$ in turn, we now add further copies of B^* to $\mathcal{B}$ in order to cover all those i-useless vertices which still lie in $A_i^\diamond$. Let U' denote the set of all these vertices. Again, each such copy of B^* will meet every $A_i^\diamond$ with $i \le q$ in precisely z vertices. It is easy to see

that these copies of B^* can be found greedily. This follows similarly as before since $|U'|$ is much smaller than the number of neighbours in any $A_j^\diamond$ of such a (non-j-exceptional) vertex $u \in U'$ and since each $u \in U'$ is joined to almost all vertices in $A_{q+1}^\diamond$. More precisely, given $u \in U' \cap A_k^\diamond$, let $i \le q$ with $i \neq k$ be such that $|N(u) \cap A_i^\diamond|$ is minimal. Note that this implies that $|N(u) \cap A_j^\diamond| \ge |A_j^\diamond|/3$ for all $j \le q$ with $j \neq i, k$. We choose the copy of B^* containing u by first picking z neighbours of u in $A_i^\diamond$ which are not i-useless (this can be done since u has at least $\tau^{1/3}|A_i^\diamond|/3$ neighbours in $A_i^\diamond$). Then we pick a copy of B_1^* in $A_{q+1}^\diamond$ which is joined to all the $z+1$ vertices chosen before and which also avoids all the useless vertices. Finally, we pick the remaining vertices.

Call a vertex $u \in A_{q+1}^\diamond$ *worthless* if u has at most $(1-3\tau^{1/4})|A_i^\diamond|$ neighbours in $A_i^\diamond$ for some $i \le q$. Thus every worthless vertex is either $(q+1)$-useless or lies in U. In particular, at most $\tau^{2/3}n$ vertices are worthless. For each worthless vertex u we will remove a copy of B^* containing u. Since u has at least $\tau^{1/3}|A_i^\diamond|/3$ neighbours in $A_i^\diamond$ for each $i \le q$, it is easy to see that this can be done if $q = \ell - 1$. So suppose that $q < \ell - 1$.

We now consider all the worthless vertices u in turn. Again, we let $i \le q$ be such that $|N(u) \cap A_i^\diamond|$ is minimal. Thus $|N(u) \cap A_j^\diamond| \ge |A_j^\diamond|/3$ for all $j \le q$ with $j \neq i$. Choose a set T_u of z neighbours of u in $A_i^\diamond$. Let N_u denote the set of all those common neighbours of the vertices from T_u in the set $A_{q+1}^\diamond$ which are not worthless. Thus

$$|N_u| \ge (1-\tau^{1/5})|A_{q+1}^\diamond|. \tag{20}$$

We will show that there are many disjoint copies of B_1^* in $G[N_u]$ such that all but one vertex class in each of these copies lie in the neighbourhood of u. We will call such a copy of B_1^* *good for u*.

Let $t := \lceil 3/\xi \rceil$. Let K^* denote the complete $(\ell - q)$-partite graph with $\ell - q - 1$ vertex classes of size zt and one vertex class of size $z_1 t$. Note that $\chi_{cr}(K^*) = \chi_{cr}(B_1^*)$. Thus Theorem 3 together with (20) and the first inequality in (19) imply that $G[N_u]$ contains a K^*-packing which covers all but at most $\tau^{1/6}|N_u|$ vertices. On the other hand, similarly as in (18) we have

$$\begin{aligned}|N(u) \cap N_u| &\overset{(20)}{\ge} \left(1 - \frac{1}{\chi_{cr}(B_1^*)} - \tau^{1/6}\right)|N_u| \overset{(3)}{=} \left(\frac{(\ell - q - 2 + \xi)zt}{(\ell - q - 1 + \xi)zt} - \tau^{1/6}\right)|N_u| \\ &\ge \left(\frac{(\ell - q - 2)zt + 2z}{|K^*|} + \tau^{1/7}\right)|N_u|,\end{aligned}$$

where the last inequality holds since $\xi zt \ge 2z + 2|K^*|\tau^{1/7}$. Thus there are many copies of K^* in the K^*-packing such that u is joined to at least z vertices in all but at most one class. Each such copy of K^* gives a copy of B_1^* which is good for u. Take such a copy of B_1^*, exchange u with an appropriate vertex, extend the new copy of B_1^* to a copy of B^* (which will meet $A_i^\diamond$ precisely in T_u) and remove it. Since there is room to spare in the calculations above, we can do this for every worthless vertex u in turn.

Let A_i^* denote the subset of all those vertices in $A_i^\diamond$ which are not covered by some copy of B^* or H in $\mathcal{B}$. Then the sets $A_1^*, \dots, A_{q+1}^*$ are as required in the lemma. □

We first deal with the case where G looks very much like the complete ℓ-partite graph whose vertex class sizes are a multiple of those of the bottlegraph B^* of H.

Lemma 22. *Suppose that H is a graph of chromatic number $\ell \ge 2$ such that $\mathrm{hcf}(H) = 1$. Let B^* denote the bottlegraph assigned to H. Let ξ, z and z_1 be as defined in (2) and let $0 < \tau \ll \xi, 1-\xi, 1/|B^*|$. Let $|B^*| \ll D \ll C$ be integers such that D is divisible by*

$2|B^*|$. *Let G be a graph whose order $n \gg C, 1/\tau$ is divisible by $|B^*|$ and which satisfies the following two properties:*

(i) $\delta(G) \geq (1 - \frac{1}{\chi_{cr}(H)})n + C$.

(ii) *The vertex set of G can be partitioned into $A_1, \dots, A_\ell$ such that, for all $i < \ell$, we have $|A_i| = (n - \ell D)z/|B^*| + D$ and $d(A_i) \leq \tau$.*

Then G has a perfect H-packing.

Proof. Our aim is to find a subgraph of G for which it is clear that we can apply the Blow-up lemma to find a perfect H-packing. We first apply Lemma 21 with $q := \ell - 1$ and $D' := D/2$ to obtain sets $A_1^*, \dots, A_\ell^*$. Let G^* denote the subgraph of G induced by the union of all these A_i^*. So $G - G^*$ has a perfect H-packing. It is easy to see (and follows from Lemma 18 applied with $D' = D/2$) that the complete ℓ-partite graph with vertex classes $A_1^*, \dots, A_\ell^*$ has a perfect H-packing. Since in G^* each vertex in A_i^* has at least $(1 - \tau^{1/5})|A_j^*|$ neighbours in each other A_j^* the bipartite subgraph of G^* between every pair A_i^*, A_j^* of sets is $(2\tau^{1/5}, 1/2)$-superregular. Hence the Blow-up lemma implies that G^* has a perfect H-packing. Together with all the copies of H chosen so far this yields a perfect H-packing in G. □

Another family of graphs having large minimum degree but not containing a perfect H-packing can be obtained from a complete ℓ-partite graph whose vertex classes are multiples of those of the bottlegraph B^* as follows: remove all edges between the smallest vertex class (A_ℓ say) and one of the others ($A_{\ell-1}$ say), remove one vertex x from A_ℓ, delete all the edges between x and A_1 and add x to A_1, add all edges within the remainder of A_ℓ and add a sufficient number of edges within $A_{\ell-1}$. The next lemma deals with the case where G is similar to this family of graphs, although slightly more dense.

Lemma 23. *Suppose that H is a graph of chromatic number $\ell \geq 2$ such that $\mathrm{hcf}(H) = 1$. Let B^* denote the bottlegraph assigned to H. Let ξ, z and z_1 be as defined in (2) and let $0 < \tau \ll \xi, 1-\xi, 1/|B^*|$. Then there exists an integer $s_0 = s_0(\tau, H)$ such that the following holds. Let $|B^*| \ll D \ll C$ be integers such that D is divisible by s_0. Let G be a graph whose order $n \gg C, 1/\tau$ is divisible by $|B^*|$ and which satisfies the following properties:*

(i) $\delta(G) \geq (1 - \frac{1}{\chi_{cr}(H)})n + C$.

(ii) *There are disjoint vertex sets $A_1, \dots, A_{\ell-2}$ in G such that $|A_i| = (n - \ell D)z/|B^*| + D$ and $d(A_i) \leq \tau$ for all $i \leq \ell - 2$.*

(iii) *The graph $G_1 := G - \bigcup_{i=1}^{\ell-2} A_i$ contains a vertex set A such that $d(A, V(G_1) \setminus A) \leq \tau$.*

Then G has a perfect H-packing.

Proof. Put
$$p := \lfloor 4/\xi \rfloor.$$
Fix further constants $\varepsilon', d', \theta, \tau_2, \dots, \tau_p$ such that
$$0 < \varepsilon' \ll d' \ll \theta \ll \tau \ll \tau_2 \ll \tau_3 \ll \dots \ll \tau_p \ll \xi, 1 - \xi, 1/|B^*|.$$
Let B_1^* be the complete bipartite graph with vertex classes of size z_1 and z (in other words, it is the subgraph of B^* induced by its two smallest vertex classes). Let $k_1(\varepsilon', \theta, B_1^*)$ be as defined in Lemma 13. Put
$$s_0 := 4k_1(p!)|B_1^*||B^*|.$$
Let $q := \ell - 2$ and $A_{q+1} := V(G_1)$. If $\ell \geq 3$ we first apply Lemma 21 with $D' = D/2$ to obtain sets $A_1^*, \dots, A_{q+1}^*$. Let G^* denote the subgraph of G induced by all the A_i^* and

put $n^* := |G^*|$. Thus G^* was obtained from G by taking out a small number of disjoint copies of H and $n - n^* \le \tau^{3/5}n$. We have to show that G^* has a perfect H-packing. Put $G_1^* := G[A_{q+1}^*]$ and $n_1^* := |G_1^*|$. Note that n_1^* is divisible by $|B_1^*|$. (In the case when $\ell = 2$ we put $G^* = G_1^* = G$.) Also, it will be crucial later on that

$$\frac{\delta(G_1^*)}{n_1^*} \ge 1 - \frac{1}{\chi_{cr}(B_1^*)} - \tau^{1/2} = \frac{\xi}{1+\xi} - \tau^{1/2}. \tag{21}$$

This can be proved in the same way as (16), the only difference is that we have to account for the fact that $V(G)$ and $V(G^*)$ are not quite the same. (But since $n - n^* \le \tau^{3/5}n$, we can compensate for this by including the error term $\tau^{1/2}$ in the above.)

Ideally, we would like to choose a perfect B_1^*-packing in G_1^* using Lemma 13. If $\ell \ge 3$ we would like to extend each copy of B_1^* to a copy of B^* by adding suitable vertices in $A_1^* \cup \dots \cup A_q^*$. Inequality (21) implies that G_1^* has sufficiently large minimum degree for this. However, we cannot apply Lemma 13 directly to G_1^* since the condition (ii) is not satisfied. So we will consider the 'almost components' of G_1^* instead.

Note that if $C' \subseteq V(G_1^*)$ is such that $d(C', V(G_1^*) \setminus C') \le \tau_p$ then $|C'| \ge \delta(G_1^*) - \tau_p n_1^* \ge \xi n_1^*/3$. Let $r \le p$ be maximal such that there is a partition $C_1, \dots, C_r$ of $V(G_1^*)$ with $d(C_j, V(G_1^*) \setminus C_j) \le \tau_r$ for all $j \le r$. We have just seen that $r \le 3/\xi < p$ and

$$\left(\frac{\xi}{1+\xi} - \tau_r^{1/2}\right) n_1^* \le |C_j| \le \left(\frac{1}{1+\xi} + \tau_r^{1/2}\right) n_1^*. \tag{22}$$

Moreover, $r \ge 2$ since $d(A \cap V(G_1^*), V(G_1^*) \setminus A) \le \tau_2$.

Recall that the aim is to choose a perfect B_1^*-packing in $G_1^*[C_j]$ (for all $j \le r$) and to extend each copy of B_1^* to a copy of B^* by adding suitable vertices in $A_1^* \cup \dots \cup A_q^*$. Our choice of r ensures that no $G_1^*[C_j]$ is close to an extremal graph and thus to find a perfect B_1^*-packing we can argue similarly as in the non-extremal case (c.f. Lemma 13 and Corollary 14).

More precisely, we proceed as follows. The first step is to tidy up the sets C_j to ensure that *every* vertex in C_j has only few neighbours in $G_1^* - C_j$. (The latter implies that the minimum degree of each graph $G_1^*[C_j]$ is about $\delta(G_1^*)$.) Given a set $C' \subseteq V(G_1^*)$, put $\overline{C}' := V(G_1^*) \setminus C'$. We say that a vertex $x \in C_j$ is *j-useless* if it has less than $\xi|C_j|/3$ neighbours in C_j. By (21) each j-useless vertex x has at least $\xi|\overline{C_j}|/3$ neighbours in $\overline{C_j}$. Since we assumed that $d(C_j, \overline{C_j}) \le \tau_r$, this implies that the number of j-useless vertices is at most $\tau_r^{3/4}|C_j|$. For all $j \le r$ we remove every j-useless vertex $x \in C_j$ and add x to some C_i which contains at least $\xi|C_i|/3$ neighbours of x. We denote by C_j' the sets thus obtained from the C_j. So every vertex in C_j' has at least $\xi|C_j'|/4$ neighbours in C_j'. Moreover, $d(C_j', \overline{C_j'}) \le \tau_r^{2/3}$.

We say that a vertex $x \in C_j'$ *j-bad* if x has at least $\tau_r^{1/6}|\overline{C_j'}|$ neighbours in $\overline{C_j'}$. Thus there exist at most $\tau_r^{1/2}|C_j'|$ such vertices. For each such vertex x in turn we take out a copy of $B_1^* = K_{z,z_1}$ from $G_1^*[C_j']$ which contains x. All these copies of B_1^* can be found greedily since each vertex in C_j' has at least $\xi|C_j'|/4$ neighbours in C_j' (and thus we can apply for instance the Erdős-Stone theorem). We denote by $\mathcal{B}_j'$ the set of all copies of B_1^* chosen for the j-bad vertices. Let C_j'' be the subset obtained from C_j' in this way. Put $n'' := |C_1'' \cup \dots \cup C_r''|$.

Our next aim is to take out a *bounded* number of copies of H to ensure that for each $j \le r$ the size of the subset thus obtained from C_j'' is divisible by $|B_1^*|$. Let $D' := D/(4r)$.

Choose integers $t_j > 0$ and a_j such that $|C''_j| = |B^*_1|t_j + a_j + 4D'$ where $\sum_{i=1}^r a_j = 0$ and $|a_j| < |B^*_1|$. Note that this implies $|B^*_1| \sum t_j = n'' - D$.

We now need to distinguish the cases when $\ell \geq 3$ and $\ell = 2$. Let us first consider the case when $\ell \geq 3$. Let G' be the graph obtained from the complete $(\ell-2)$-partite graph with vertex classes of size $D/4$ by adding r complete bipartite graphs $K_{D'+a_j,D'}$ (with $1 \leq j \leq r$) and joining all the vertices of these bipartite graphs to all the vertices of the complete $(\ell-2)$-partite graph. So $|G'| = \ell D/4$. An r-fold application of Lemma 15 shows that G' contains a perfect H-packing $\mathcal{H}'$. This in turn implies that we can greedily take out $|\mathcal{H}'| = \ell D/(4|H|)$ disjoint copies of H from G to achieve that the subsets $A^\diamond_i$ and $C^\diamond_j$ thus obtained from the sets A^*_i and C''_j have the following sizes (where $n^\diamond = n^* - \ell D/4$ denotes the remaining number of vertices in the graph):

$$|A^\diamond_i| = |A^*_i| - D/4 \quad \text{and thus} \quad |A^\diamond_i| = z(n^\diamond - \ell D/4)/|B^*| + D/4 \tag{23}$$

and

$$|C^\diamond_j| = |C''_j| - 2D' - a_j = |B^*_1|t_j + D/(2r). \tag{24}$$

Note that every vertex in $C^\diamond_j$ still has at most $2\tau_r^{1/6}|\overline{C^\diamond_j}|$ neighbours in $\overline{C^\diamond_j}$. Thus

$$\delta(G^*_1[C^\diamond_j]) \overset{(21)}{\geq} \left(\frac{\xi}{1+\xi} - \tau^{1/2}\right) n^*_1 - 2\tau_r^{1/6} n^*_1 \overset{(22)}{\geq} (\xi - \tau_r^{1/7})|C^\diamond_j| \overset{(3)}{>} \left(1 - \frac{1}{\chi_{cr}(B^*_1)}\right)|C^\diamond_j| \tag{25}$$

and

$$d(C^\diamond_j, \overline{C^\diamond_j}) \leq \tau_r^{1/7}. \tag{26}$$

The arguments in the case when $\ell = 2$ are similar except that now we consider the graph G' consisting of r complete subgraphs of sizes $2D' + a_j$ (where $j = 1, \dots, r$). An $(r-1)$-fold application of Lemma 17 shows that G' has a perfect H-packing. So we can proceed similarly as before to obtain sets $C^\diamond_j$ which satisfy (24)–(26).

In order to show that $G^*_1[C^\diamond_j]$ has a perfect B^*_1-packing we wish to apply Lemma 13 with $\tau_{r+1}/2$ playing the role of τ to find a blown-up B^*_1-cover. (We will then use the fact that $\mathrm{hcf}(H) = 1$ and Lemmas 18 and 19 to find a suitable perfect B^*_1-packing of $G^*_1[C^\diamond_j]$ which is extendable to a perfect H-packing of the whole graph.) Inequality (25) shows that $G^*_1[C^\diamond_j]$ satisfies the requirement on the minimum degree in Lemma 13. We will now check that $G^*_1[C^\diamond_j]$ also satisfies the conditions (i) and (ii) there. So suppose that $G^*_1[C^\diamond_j]$ does not satisfy (ii) and let $C' \subseteq C^\diamond_j$ be such that $d(C', C^\diamond_j \setminus C') \leq \tau_{r+1}/2$. Using that $|C_i \setminus C^\diamond_i| \ll \tau_{r+1}|C_i|$ for all $i \leq r$ it is easy to see that the $r+1$ sets $C' \cap C_j$, $C_j \setminus C'$, C_i $(i \neq j)$ would then contradict the choice of r. Thus $G^*_1[C^\diamond_j]$ satisfies (ii). Suppose next that $G^*_1[C^\diamond_j]$ does not satisfy (i). Let $C' \subseteq C^\diamond_j$ be such that $|C'| = z|C^\diamond_j|/|B^*_1|$ and $d(C') \leq \tau_{r+1}/2$. Let $x \in C'$ by any vertex which has at most $\tau_{r+1}|C'|$ neighbours in C'. Then

$$d_{G^*_1[C^\diamond_j]}(x) \leq \tau_{r+1}|C'| + |C^\diamond_j \setminus C'| \leq \left(\tau_{r+1} + 1 - \frac{z}{|B^*_1|}\right)|C^\diamond_j| \overset{(25)}{<} \delta(G^*_1[C^\diamond_j]),$$

a contradiction. (In the final inequality, we also used the fact that $1 - z/|B^*_1| = \xi/(1+\xi)$.) Thus we can apply Lemma 13 to $G^*_1[C^\diamond_j]$ to obtain a B^*_1-packing $\mathcal{B}^*_j$ in $G^*_1[C^\diamond_j]$ such that the graph $G^\diamond_j := G^*_1[C^\diamond_j] - \bigcup \mathcal{B}^*_j$ (which is obtained by removing all those vertices which lie in a copy of B^*_1 in $\mathcal{B}^*_j$) has a blown-up B^*_1-cover with parameters $2\varepsilon', d'/2, 2\theta, k := k_1(\varepsilon', \theta, B^*_1)$. Note that this blown-up B^*_1-cover does not necessarily yield a perfect B^*_1-packing of $G^\diamond_j$ as we need not have $\mathrm{hcf}(B^*_1) = 1$. This will cause difficulties later on.

Recall that, when removing the j-bad vertices from C'_j, we have already set aside a B^*_1-packing $\mathcal{B}'_j$. For each $B \in \mathcal{B}^*_j \cup \mathcal{B}'_j$ we now greedily choose z vertices in each of $A^\diamond_1, \dots, A^\diamond_q$ such that all these vertices form a copy of B^* together with B. We remove these copies of B^* and still denote the subsets of the $A^\diamond_i$ obtained in this way by $A^\diamond_i$. Also, we still denote the remaining number of vertices in G by $n^\diamond$. Then it is easy to see that the second equation in (23) still holds for all i.

Consider the blown-up B^*_1-cover of $G^\diamond_j$. Let $\{X^j_i(t) \mid 1 \le t \le k,\, i = 1,2\}$ be a partition of $V(G^\diamond_j)$ as in the definition of a blown-up B^*_1-cover (Definition 11). For all $j \le r$ and all $t \le k$ we would like to apply the Blow-up lemma in order to find a B^*_1-packing which covers precisely the vertices in $X^j_1(t) \cup X^j_2(t)$. To be able to do this, we need that the complete bipartite graph with vertex classes $X^j_1(t)$ and $X^j_2(t)$ contains a perfect B^*_1-packing. Clearly, the latter is the case if $|X^j_1(t)| = z|X^j_1(t) \cup X^j_2(t)|/|B^*_1|$ and $|X^j_2(t)| = z_1|X^j_1(t) \cup X^j_2(t)|/|B^*_1|$. We will now show that this can be achieved by taking out a small number of further copies of H from G. (Note that this would be much simpler to achieve if we could assume that $\mathrm{hcf}(B^*_1) = 1$.)

Put $D'' := D/(4rk)$ and $x^j(t) := (|X^j_1(t) \cup X^j_2(t)| - 2D'')/|B^*_1|$. If $\ell \ge 3$ consider a random partition of $A^\diamond_i$ into kr sets $A^{\diamond j}_i(t)$ ($j \le r$, $t \le k$) such that $|A^{\diamond j}_i(t)| = zx^j(t) + D''$. (It is straightforward to check that these numbers sum up to exactly $|A^\diamond_i|$.) Consider the complete ℓ-partite graph $G_j(t)$ with vertex classes $A^{\diamond j}_1(t), \dots, A^{\diamond j}_{\ell-2}(t), X^j_1(t), X^j_2(t)$. As is easily seen, the sizes of these classes satisfy the conditions of Lemma 19 with D'' playing the role of D' in that lemma. Thus we can apply Lemma 19 and then Lemma 18 to obtain a subgraph $\tilde{G}_j(t)$ of $G_j(t)$ which is obtained by removing a few copies of H as described there. So this gives us a collection $\tilde{\mathcal{H}}_j(t)$ of at most $\theta^{1/20}|G_j(t)|$ disjoint copies of H in $G_j(t)$ such that the subsets $\tilde{A}^j_1(t), \dots, \tilde{A}^j_{\ell-2}(t), \tilde{X}^j_1(t), \tilde{X}^j_2(t)$ obtained from $A^{\diamond j}_1(t), \dots, A^{\diamond j}_{\ell-2}(t), X^j_1(t), X^j_2(t)$ by deleting all those vertices which lie in some copy of H in $\tilde{\mathcal{H}}_j(t)$ satisfy

$$|\tilde{A}^j_i(t)| = z\tilde{n}_j/|B^*| = |\tilde{X}^j_1(t)|$$

for all $i \le \ell - 2$ and

$$|\tilde{X}^j_2(t)| = \xi z\tilde{n}_j/|B^*|,$$

where $\tilde{n}_j := |\tilde{G}_j(t)|$. (We get the bound of $\theta^{1/20}|G_j(t)|$ copies by observing that we can apply Lemma 19 with $a_{\ell-1} \le \theta^{1/15}|G_j(t)|$ from Definition 11 and $a_i = 0$ for $i \le \ell - 2$.)

For each copy $H' \in \tilde{\mathcal{H}}_j(t)$ of H in turn we greedily remove a copy of H in $G^\diamond_j \subseteq G$ which intersects the sets $A^{\diamond j}_1(t), \dots, A^{\diamond j}_{\ell-2}(t), X^j_1(t), X^j_2(t)$ in the same way as H'. (Note that we are able to do this as Lemma 21(iii) and the fact we considered a random partition of the $A^\diamond_i$ imply that all vertices in $G^\diamond_j$ are adjacent to almost all vertices in each $A^{\diamond j}_i(t)$. Similarly, all vertices in $A^{\diamond j}_i(t)$ are adjacent to almost all vertices in each other $A^{\diamond j}_{i'}(t)$. Moreover, the pair $(X^j_1(t), X^j_2(t))$ is $(2\varepsilon', d'/2)$-superregular. This enables us to construct the required number of copies of H in $G^\diamond_j$ if we begin the construction of each copy with the vertices that lie in $X^j_1(t) \cup X^j_2(t)$. Since $\theta \gg d'$ we have to be careful that we do not destroy the superregularity of the leftover subsets of the sets $X^j_i(t)$ in this process. We can get around this difficulty by considering a random red-blue partition of the vertices as described in Section 5.1 again and removing only copies of H whose vertices are all blue.) We think of $\tilde{A}^j_1(t), \dots, \tilde{A}^j_{\ell-2}(t), \tilde{X}^j_1(t), \tilde{X}^j_2(t)$ as the subsets obtained from $A^{\diamond j}_1(t), \dots, A^{\diamond j}_{\ell-2}(t), X^j_1(i), X^j_2(t)$ in this way. We can now apply the Blow-up lemma to find a B^*_1-packing which covers precisely the vertices in $\tilde{X}^j_1(t) \cup \tilde{X}^j_2(t)$. The union of

all these B_1^*-packings over all $t \le k$ and all $j \le r$ forms a B_1^*-packing $\mathcal{B}_1$ which covers precisely the leftover vertices of the graphs $G_j^\diamond$ with $j \le r$. If $\ell = 2$ then $\mathcal{B}_1$ together with all the copies of $B_1^* = B^*$ and H chosen earlier yields a perfect H-packing of G.

If $\ell \ge 3$ then our aim is to extend $\mathcal{B}_1$ to a B^*-packing by adding all the remaining vertices in the sets $A_i^\diamond$. So for all $i \le \ell - 2$ let A_i' be the subset of $A_i^\diamond$ which is left over after removing the copies of H in the union (over all j and t) of the sets $\tilde{\mathcal{H}}_j(t)$ described above. In order to extend $\mathcal{B}_1$ to a B^*-packing which also covers the vertices in the sets A_i', we consider the following $(\ell-1)$-partite auxiliary graph J. The vertex classes of J are $A_1', \dots, A_{\ell-2}', \mathcal{B}_1$. The subgraph of J induced by $A_1', \dots, A_{\ell-2}'$ is nothing else than the $(\ell-2)$-partite subgraph of G induced by these sets. J contains an edge between $x \in A_j'$ and $B_1^* \in \mathcal{B}_1$ if x is joined (in G) to all vertices of B_1^*. Using Lemma 21(iii) and the fact that we deleted only comparatively few vertices so far, it is easy to check that in each of the $\binom{\ell-1}{2}$ bipartite subgraphs forming J, every vertex is adjacent to all but a τ_{r+1}-fraction of the vertices in the other class (with room to spare). Thus the bipartite subgraphs are all $(2\tau_{r+1}, 1/2)$-superregular. Let B_2^* be the complete $(\ell-1)$-partite graph with $\ell - 2$ vertex classes of size z and one vertex class of size 1. The Blow-up lemma implies that J has a perfect B_2^*-packing. This corresponds to a B^*-packing (and thus also an H-packing) in G. Together with all the copies of H chosen earlier this yields a perfect H-packing in G. □

The final lemma in this section deals with the remaining 'extremal' possibilities: so G contains at least one large almost independent set A but does not satisfy the conditions of either of the two previous lemmas.

Lemma 24. *Suppose that H is a graph of chromatic number $\ell \ge 3$ such that $\mathrm{hcf}(H) = 1$. Let B^* denote the bottlegraph assigned to H. Let ξ, z and z_1 be as defined in (2) and let $0 < \tau \ll \tau' \ll \xi, 1-\xi, 1/|B^*|$. Then there exists an integer $s_1 = s_1(\tau, \tau', H)$ such that the following holds. Let $|B^*| \ll D \ll C$ and $1 \le q \le \ell - 2$ be integers such that D is divisible by s_1. Let G be a graph whose order $n \gg C, 1/\tau$ is divisible by $|B^*|$ and which satisfies the following properties:*

- (i) $\delta(G) \ge (1 - \frac{1}{\chi_{cr}(H)})n + C$.
- (ii) *There are disjoint vertex sets $A_1, \dots, A_q$ in G such that $|A_i| = (n - \ell D)z/|B^*| + D$ and $d(A_i) \le \tau$ for all $i \le q$.*
- (iii) *G does not contain disjoint vertex sets $A_1', \dots, A_{q+1}'$ such that $|A_i'| = (n - \ell D)z/|B^*| + D$ and $d(A_i') \le \tau'$ for all $i \le q + 1$.*
- (iv) *If $q = \ell - 2$, then the graph $G_1 := G - \bigcup_{i=1}^{\ell-2} A_i$ contains no vertex set A so that $d(A, V(G_1) \setminus A) \le \tau'$.*

Then G has a perfect H-packing.

Proof. Let $\theta := \tau^{1/2}$. Fix further constants ε', d' such that

$$0 < \varepsilon' \ll d' \ll \tau \ll \tau' \ll \xi, 1 - \xi, 1/|B^*|.$$

Let B_1^* denote the complete $(\ell - q)$-partite graph with $\ell - q - 1$ vertex classes of size z and one vertex class of size z_1. Let $k_1(\varepsilon', \theta, B_1^*)$ be as defined in Lemma 13. Put

$$s_1 := 2k_1|B_1^*||B^*|.$$

Let $A_{q+1} := V(G) \setminus \bigcup_{i=1}^{q} A_i$. As in the proof of Lemma 23, we first apply Lemma 21 with $D' = D/2$ to obtain sets $A_1^*, \dots, A_{q+1}^*$. Let G^* denote the subgraph of G induced by all the A_i^* and put $n^* := |G^*|$. Thus G^* was obtained from G by taking out a small number

of disjoint copies of H and $n - n^* \le \tau^{3/5} n$. Put $G_1^* := G[A_{q+1}^*]$ and $n_1^* := |G_1^*|$. Note that n_1^* is divisible by $|B_1^*|$. Moreover, as in (16) or (21) one can show that

$$\delta(G_1^*) \ge \left(1 - \frac{1}{\chi_{cr}(B_1^*)} - \tau^{1/2}\right) n_1^*. \tag{27}$$

Similarly as in Lemma 23, to find a perfect H-packing in G^* our aim is to choose a perfect B_1^*-packing in G_1^* and extend each copy of B_1^* to a copy of B^* by adding suitable vertices in $A_1^* \cup \dots \cup A_q^*$. Again, we wish to apply Lemma 13 with $\tau'/2$ playing the role of τ to G_1^* in order to do this. Thus we have to check that G_1^* satisfies the conditions of Lemma 13. Inequality (27) implies that G_1^* satisfies the condition on the minimum degree. Suppose that G_1^* does not satisfy condition (i) of Lemma 13. So there exists a set $A' \subseteq V(G_1^*)$ such that $|A'| = zn_1^*/|B_1^*|$ and $d(A') \le \tau'/2$. It is easy to check that $(1 - \tau^{1/2})|A_1| \le |A'| \le (1 + \tau^{1/2})|A_1|$. Thus by changing a small number of vertices we obtain a set $A'' \subseteq V(G) \setminus (A_1 \cup \dots \cup A_q)$ such that $d(A'') \le 2\tau^{1/2} + \tau'/2 \le \tau'$. But then the sets $A_1, \dots, A_q, A''$ contradict condition (iii) in Lemma 24. So G_1^* satisfies condition (i) of Lemma 13. Next suppose that $q = \ell - 2$ and G_1^* does not satisfy condition (ii) of Lemma 13. So G_1^* contains a set A with $d(A, V(G_1^*) \setminus A) \le \tau'/2$. Since $V(G_1)$ and $V(G_1^*)$ are almost the same, this in turn implies that the graph G_1 defined in (iv) contains a set A with $d(A, V(G_1) \setminus A) \le 2\tau'/3$, a contradiction to the assumption in (iv). Thus we may assume that G_1^* also satisfies condition (ii) of Lemma 13. So we can apply Lemma 13 to G_1^*. This shows that we can take out a small number of copies of B_1^* to obtain a subgraph of G_1^* which has a blown-up B_1^*-cover. We can then proceed similarly as in the final part of the proof of Lemma 23. □

8. Proof of Theorem 4

We will now combine the results of Sections 5 and 7 to prove Theorem 4. Fix constants

$$0 < \varepsilon' \ll d' \ll \theta \ll \tau_1 \ll \dots \ll \tau_{\ell-1} \ll \xi, 1 - \xi, 1/|B^*|.$$

Let $D \gg |B^*|$ be an integer satisfying the conditions in Lemmas 22–24. Let $C \gg D, 1/\tau$. By taking out at most $\ell - 2$ disjoint copies of H from G if necessary, we may assume that the order n of our given graph G is divisible by $|B^*|$. (The existence of such copies follows from the Erdős-Stone-theorem.) We are done if G satisfies conditions (i) and (ii) of Corollary 14 with τ_1 playing the role of τ.

So suppose first that G violates (i). Thus there is some set $A \subseteq V(G)$ of size $zn/|B^*|$ such that $d(A) \le \tau_1$. Choose $q \le \ell - 1$ maximal such that there are disjoint sets $A_1', \dots, A_q' \subseteq V(G)$ with $|A_i'| = zn/|B^*|$ and $d(A_i') \le \tau_q$ for all $i \le q$. So $q \ge 1$ by our assumption. By removing a constant number of vertices from each A_i' we obtain subsets A_i with $|A_i| = z(n - \ell D)/|B^*| + D$ and $d(A_i) \le 2\tau_q$. If $q = \ell - 1$ then Lemma 22 shows that G has a perfect H-packing. Thus we may assume that $q \le \ell - 2$. But then we can apply either Lemma 23 with τ_{q+1} playing the role of τ or Lemma 24 with $2\tau_q$ playing the role of τ and τ_{q+1} playing the role of τ'.

If G satisfies condition (i) in Corollary 14 but violates (ii) then we are done by Lemma 23 applied with $\tau := \tau_1$. This completes the proof of Theorem 4. □

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Daniela Kühn & Deryk Osthus
School of Mathematics
Birmingham University
Edgbaston
Birmingham B15 2TT
UK
E-mail addresses: {kuehn,osthus}@maths.bham.ac.uk